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Derived Limits in Quasi-Abelian Categories

by

Fabienne Prosmans

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Laboratoire Analyse, Géométrie et Applications, UMR 7539
Institut Galilée, Université Paris-Nord
93430 Villetaneuse (France)

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Abstract

In this paper, we study the derived functors of projective limit functors in quasi-abelian categories. First, we show that if $\mathcal{E}$ is a quasi-abelian category with exact products, projective limit functors are right derivable and their derived functors are computable using a generalization of a construction of Roos. Next, we study index restriction and extension functors and link them trough the symbolic Hom-functor. If $J : \mathcal{J} \longrightarrow \mathcal{I}$ is a functor between small categories and if E is a projective system indexed by $\mathcal{I}$, this allows us to give a condition for the derived projective limits of E and $E \circ J$ to be isomorphic. Note that this condition holds, if $\mathcal{I}$ and $\mathcal{J}$ are filtering and J is cofinal. Using the preceding results, we establish that the n-th left cohomological functor of the derived projective limit of a projective system indexed by $\mathcal{I}$ vanishes for $n \geq k$, if the cofinality of $\mathcal{I}$ is strictly lower than the k-th infinite cardinal number. Finally, we consider the limits of pro-objects of a quasi-abelian category. From our study, it follows, in particular, that the derived projective limit of a filtering projective system depends only on the associated pro-object.

Contents

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0 Introduction

It is well-known that the projective limit of a short exact sequence of projective systems of abelian groups is not always an exact sequence. This phenomenon often explains the problems one meets in the globalization of local results in algebra or analysis. To understand this loss of exactness, it is natural to study the derived functors of the projective limit functor. This was done in the sixties by various authors ([3, 4, 6, 8], etc.) and led to a rather good understanding of the homological algebra of projective limits in abelian categories. However, for various applications to algebraic analysis, it would be very useful to extend these results to non-abelian categories such as the category of topological abelian groups or the category of locally convex topological vector spaces. This is what we do in this paper in the more general framework of quasi-abelian categories.

We start with a summary of the facts about the homological algebra of quasi-abelian categories which are needed in the other sections (see [9] and [7] for more detailed expositions). This should help the reader which has a good knowledge of the language of homological algebra and derived categories in the abelian case (as exposed e.g. in [5]) to understand the rest of the paper. First, we recall the conditions a category $\mathcal{E}$ has to satisfy to be quasi-abelian. Next, we explain briefly the construction of the derived category $D(\mathcal{E})$ and we give the main results about the two canonical t-structures on $D(\mathcal{E})$ and their corresponding hearts $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$. We end this section by recalling how to derive an additive functor between two quasi-abelian categories.

Section 2 is devoted to the study of the category of projective systems in a quasi-abelian category $\mathcal{E}$. We show first that they form a quasi-abelian category. Then, using projective systems of product type, we prove that this category has enough injective objects when $\mathcal{E}$ itself has enough injective objects. We conclude by defining the index restriction and extension functors.

In Section 3, we explain how to derive the projective limit functor

$$\varprojlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

where $\mathcal{E}$ is a quasi-abelian category and $\mathcal{I}$ a small category. First, we consider the easy case where $\mathcal{E}$ has enough injective objects. Next, we treat the case where $\mathcal{E}$ has exact products. In this case, we show that the derived projective limit of a projective system of $\mathcal{E}$ is isomorphic to its Roos complex. Moreover, if $J : \mathcal{J} \longrightarrow \mathcal{I}$ is a functor between two small categories and E is a projective system of $\mathcal{E}$ indexed by $\mathcal{I}$, we show how to compute the canonical morphism

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} E(J(j))$$

by means of Roos complexes. By duality, we get corresponding results for the inductive limits. At the end of this section, we establish commutation formulas for derived limits and the derived Hom functor.

In Section 4, we recall a few properties of symbolic-Hom functors and show how to derive them. Then, we prove that derived projective limits may be computed using suitable derived symbolic-Hom functors. This allows us to give a condition for the canonical morphism

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} E(J(j))$$

to be an isomorphism.

In the first part of Section 5, using the preceding result, we show that if a functor $J : \mathcal{J} \longrightarrow \mathcal{I}$ between small filtering categories is cofinal, then

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \mathrm{R}\varprojlim_{j\in\mathcal{J}} E(J(j))$$

in $D^+(\mathcal{E})$. In the second part, we establish that, if the cofinality of the small filtering category $\mathcal{I}$ is strictly lower than the k-th infinite cardinal number,

$$LH^n(\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i)) = 0 \qquad \forall\, n \geq k$$

for any projective system E of $\mathcal{E}$ indexed by $\mathcal{I}$.

Section 6 is devoted to a review of the results about pro-objects we need in Section 7. We refer the reader to [2] for details. In the first three parts of this section, we recall basic results about pro-objects, representable functors and representation of diagrams of pro-objects. In the last part, we show that the category of pro-objects of an arbitrary category has filtering projective limits and we establish some properties of these limits.

In Section 7, we prove that the category of pro-objects of a quasi-abelian category is also quasi-abelian and has exact products. Next, we introduce the functor

$$\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \longrightarrow \mathcal{E}$$

and we establish that if the category $\mathcal{E}$ is quasi-abelian and has exact products, then the functor L is right derivable and

$$\mathrm{RL} \circ \text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i) \simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i)$$

for any filtering projective system E indexed by $\mathcal{I}$. This shows in particular that the derived projective limit of a filtering projective system depends only on the associated pro-object.

Note that the theory developed in this paper may be applied to the category of topological abelian groups or the category of locally convex topological vector spaces since these categories are quasi-abelian and have exact products. In these cases, more specific results may be obtained. Work in this direction is in progress and will appear elsewhere.

To conclude this introduction, I want to thank J.-P. Schneiders for the useful discussions we had during the preparation of this paper.

1 Quasi-abelian homological algebra

1.1 Quasi-abelian categories

To avoid confusions, let us first recall a few basic definitions.

Definition 1.1.1. Let $\mathcal{A}$ be an additive category and let $f : A \longrightarrow B$ be a morphism of $\mathcal{A}$.

(i) A *kernel* of f is the data of a pair $(\ker f, i)$ where $\ker f \in \mathrm{Ob}(\mathcal{A})$ and $i \in \mathrm{Hom}_{\mathcal{A}}(\ker f, A)$ are such that $f \circ i = 0$ and for any $g \in \mathrm{Hom}_{\mathcal{A}}(C, A)$ verifying $f \circ g = 0$, there is a unique $g' \in \mathrm{Hom}_{\mathcal{A}}(C, \ker f)$ making the diagram

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & A \arrow[r, "f"] & B \\
& C \arrow[ul, "g'"] \arrow[u, "g"] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutative.

(ii) A *cokernel* of f is the data of a pair $(\operatorname{coker} f, q)$ where $\operatorname{coker} f \in \mathrm{Ob}(\mathcal{A})$ and $q \in \mathrm{Hom}_{\mathcal{A}}(B, \operatorname{coker} f)$ are such that $q \circ f = 0$ and for any $g \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ verifying $g \circ f = 0$, there is a unique $g' \in \mathrm{Hom}_{\mathcal{A}}(\operatorname{coker} f, C)$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "q"] \arrow[d, "g"] & \operatorname{coker} f \arrow[dl, "g'"] \\
& C &
\end{tikzcd}$$

commutative.

One can check that $i : \ker f \longrightarrow A$ is monomorphic and that $q : B \longrightarrow \operatorname{coker} f$ is epimorphic. Moreover, two kernels (resp. two cokernels) of f are canonically isomorphic.

(iii) If the morphism $i : \ker f \longrightarrow A$ has a cokernel, it is called the *coimage* of f and denoted by $\operatorname{coim} f$.

(iv) If the morphism $q : B \longrightarrow \operatorname{coker} f$ has a kernel, it is called the *image* of f and denoted by $\operatorname{im} f$.

Remark 1.1.2. There is a canonical morphism

$$\operatorname{coim} f \longrightarrow \operatorname{im} f.$$

As a matter of fact, since $\operatorname{coim} f$ is the cokernel of $i : \ker f \longrightarrow A$, there is a unique morphism $f' : \operatorname{coim} f \longrightarrow B$ making the diagram

$$\begin{tikzcd}
\ker f \arrow[r, "i"] \arrow[dr, "0"'] & A \arrow[r, "q'"] \arrow[d, "f"] & \operatorname{coim} f \arrow[dl, "f'"] \\
& B &
\end{tikzcd}$$

commutative. We have $q \circ f' \circ q' = q \circ f = 0$. The morphism q' being epimorphic, it follows that $q \circ f' = 0$. Since $\operatorname{im} f$ is the kernel of $q : B \longrightarrow \operatorname{coker} f$, there is a unique morphism $\operatorname{coim} f \longrightarrow \operatorname{im} f$ making the diagram

$$\begin{tikzcd}
\operatorname{im} f \arrow[r, "i'"] & B \arrow[r, "q"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[ul, dashed] \arrow[u, "f'"] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutative.

Definition 1.1.3. A category $\mathcal{E}$ is *quasi-abelian* if

(i) it is additive,

(ii) any morphism has a kernel and a cokernel,

(iii) in a cartesian square

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X' \arrow[u] \arrow[r, "f'"'] & Y' \arrow[u]
\end{tikzcd}$$

f is a strict epimorphism, then f' is a strict epimorphism,

(iv) in a cocartesian square

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u] \arrow[r, "f"'] & Y \arrow[u]
\end{tikzcd}$$

f is a strict monomorphism, then f' is a strict monomorphism.

1.2 Derivation of a quasi-abelian category

In this section, $\mathcal{E}$ will be a quasi-abelian category. We denote by $C(\mathcal{E})$ the category of complexes of $\mathcal{E}$ and by $K(\mathcal{E})$ the category defined by

$$\mathrm{Ob}(K(\mathcal{E})) = \mathrm{Ob}(C(\mathcal{E}))$$

and

$$\mathrm{Hom}_{K(\mathcal{E})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{C(\mathcal{E})}(X^{\cdot}, Y^{\cdot})/Ht(X^{\cdot}, Y^{\cdot})$$

where

$$Ht(X^{\cdot}, Y^{\cdot}) = \{f^{\cdot} : X^{\cdot} \longrightarrow Y^{\cdot} : f^{\cdot} \text{ is homotopic to zero}\}.$$

Recall that $K(\mathcal{E})$ has a canonical structure of triangulated category.

Definition 1.2.1. A *sequence*

$$A \xrightarrow{u} B \xrightarrow{v} C$$

of $\mathcal{E}$ such that $v \circ u = 0$ is *strictly exact* if u is strict and if the canonical morphism

$$\operatorname{im} u \longrightarrow \ker v$$

is isomorphic.

Remark 1.2.2. Note that a sequence

$$E \longrightarrow F \longrightarrow G$$

of $\mathcal{E}$ such that

$$\mathrm{Hom}\,(X, E) \longrightarrow \mathrm{Hom}\,(X, F) \longrightarrow \mathrm{Hom}\,(X, G)$$

is exact for any $X \in \mathrm{Ob}(\mathcal{E})$ is strictly exact.

Definition 1.2.3.
(i) A *complex* $X^{\cdot}$ of $\mathcal{E}$ is *strictly exact in degree* k if the sequence

$$X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1}$$

is strictly exact.

(ii) A *complex* of $\mathcal{E}$ is *strictly exact* if it is strictly exact in every degree.

(iii) We denote by $\mathcal{N}(\mathcal{E})$ the full subcategory of $K(\mathcal{E})$ whose objects are the strictly exact complexes of $\mathcal{E}$.

(iv) A morphism $f^\cdot : X^\cdot \to Y^\cdot$ of $K(\mathcal{E})$ is called a *strict quasi-isomorphism* if there is a distinguished triangle

$$X^\cdot \to Y^\cdot \to Z^\cdot \to X^\cdot[1]$$

of $K(\mathcal{E})$ such that $Z^\cdot \in \mathrm{Ob}(\mathcal{N}(\mathcal{E}))$.

Proposition 1.2.4. *The subcategory $\mathcal{N}(\mathcal{E})$ of $K(\mathcal{E})$ is a null system, i.e. it verifies the following conditions:*

(i) $0 \in \mathcal{N}(\mathcal{E})$,

(ii) $X^\cdot \in \mathcal{N}(\mathcal{E})$ *if and only if* $X^\cdot[1] \in \mathcal{N}(\mathcal{E})$,

(iii) if

$$X^\cdot \to Y^\cdot \to Z^\cdot \to X^\cdot[1]$$

is a distinguished triangle of $K(\mathcal{E})$ where $X^\cdot$, $Y^\cdot \in \mathrm{Ob}(\mathcal{N}(\mathcal{E}))$, then $Z^\cdot \in \mathrm{Ob}(\mathcal{N}(\mathcal{E}))$.

Definition 1.2.5. The *derived category* of $\mathcal{E}$ denoted by $D(\mathcal{E})$ is the localization of the triangulated category $K(\mathcal{E})$ by $\mathcal{N}(\mathcal{E})$. Then,

$$D(\mathcal{E}) = K(\mathcal{E})/\mathcal{N}(\mathcal{E}).$$

Remark 1.2.6. Note that as in the abelian case, a strictly exact sequence

$$0 \to X^\cdot \to Y^\cdot \to Z^\cdot \to 0$$

of $C(\mathcal{E})$ gives rise to a distinguished triangle

$$X^\cdot \to Y^\cdot \to Z^\cdot \to X^\cdot[1]$$

of $D(\mathcal{E})$.

1.3 t-structure and heart of the derived category

First, let us recall some usual results about t-structures on a triangulated category.

Definition 1.3.1. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ be two strictly full subcategories of $\mathcal{T}$. We set

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n] \qquad \text{and} \qquad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n].$$

Then, the pairs $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ forms a *t-structure* on $\mathcal{T}$ if it verifies the following conditions:

(i) $\mathcal{T}^{\leq -1} \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$,

(ii) $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ if $X \in \mathrm{Ob}(\mathcal{T}^{\leq 0})$ and $Y \in \mathrm{Ob}(\mathcal{T}^{\geq 1})$,

(iii) for any $X \in \mathrm{Ob}(\mathcal{T})$, there is a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1]$$

such that $X_0 \in \mathrm{Ob}(\mathcal{T}^{\leq 0})$ and $X_1 \in \mathrm{Ob}(\mathcal{T}^{\geq 1})$.

The *heart of the t-structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, denoted by $\mathcal{H}$, is the full subcategory of $\mathcal{T}$ defined by

$$\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}.$$

Theorem 1.3.2. *The heart of any t-structure is an abelian category.*

Proposition 1.3.3. *Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{T}$.*

(i) There is a functor

$$\tau^{\leq n} : \mathcal{T} \longrightarrow \mathcal{T}^{\leq n}$$

such that

$$\mathrm{Hom}_{\mathcal{T}^{\leq n}}(X, \tau^{\leq n}(Y)) \simeq \mathrm{Hom}_{\mathcal{T}}(X, Y)$$

for any $X \in \mathrm{Ob}(\mathcal{T}^{\leq n})$ and any $Y \in \mathrm{Ob}(\mathcal{T})$. In the same way, there is a functor

$$\tau^{\geq n} : \mathcal{T} \longrightarrow \mathcal{T}^{\geq n}$$

such that

$$\mathrm{Hom}_{\mathcal{T}^{\geq n}}(\tau^{\geq n}(X), Y) \simeq \mathrm{Hom}_{\mathcal{T}}(X, Y)$$

for any $X \in \mathrm{Ob}(\mathcal{T})$ and $Y \in \mathrm{Ob}(\mathcal{T}^{\geq n})$. These functors $\tau^{\leq n}$ and $\tau^{\geq n}$ are the truncation functors *associated to the t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.*

(ii) For any $n \in \mathbb{Z}$, the functor

$$H^n : \mathcal{T} \longrightarrow \mathcal{H}$$

defined by

$$H^n(X) = (\tau^{\leq n} \circ \tau^{\geq n}(X))[n]$$

is a cohomological functor, i.e. any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

of $\mathcal{T}$ gives rise to the long exact sequence

$$\begin{tikzcd} \cdots & H^n(X) \arrow[r] & H^n(Y) \arrow[r] & H^n(Z) \arrow[dll, rounded corners] \\ & H^{n+1}(X) \arrow[r] & H^{n+1}(Y) \arrow[r] & H^{n+1}(Z) & \cdots \end{tikzcd}$$

Definition 1.3.4. Let $\mathcal{E}$ be a quasi-abelian category. We denote by

$$D^{\leq 0}(\mathcal{E}) \qquad (\text{resp. } D^{\geq 0}(\mathcal{E}))$$

the full subcategory of $D(\mathcal{E})$ whose objects are the complexes which are strictly exact in degree $k > 0$ (resp. $k < 0$).

Proposition 1.3.5. *Let $\mathcal{E}$ be a quasi-abelian category. The pair $(D^{\leq 0}, D^{\geq 0})$ forms a t-structure on $D(\mathcal{E})$. We call it the* left t-structure *of $D(\mathcal{E})$.*

Remark 1.3.6. The heart of the left t-structure is denoted by

$$\mathcal{LH}(\mathcal{E}) = D^{\leq 0}(\mathcal{E}) \cap D^{\geq 0}(\mathcal{E}).$$

We call it the *left heart* of $D(\mathcal{E})$. Of course, the objects of $\mathcal{LH}(\mathcal{E})$ are the complexes which are strictly exact in every degree but zero.

The cohomological functors are denoted by

$$LH^k : D(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{E}).$$

Proposition 1.3.7. *Let $\mathcal{E}$ be a quasi-abelian category. Let $X^\cdot$ be an object of $D(\mathcal{E})$. The truncation functors are given by*

$$\tau^{\leq n}(X^\cdot) : \cdots \longrightarrow X^{n-1} \longrightarrow \ker d^n \longrightarrow 0$$

where $\ker d^n$ is in degree n and

$$\tau^{\geq n}(X^\cdot) : 0 \longrightarrow \operatorname{coim} d^{n-1} \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \cdots$$

where X^n is in degree n. Hence, the cohomological functors are given by

$$LH^n(X^\cdot) : 0 \longrightarrow \operatorname{coim} d^{n-1} \longrightarrow \ker d^n \longrightarrow 0$$

where $\ker d^n$ is in degree 0.

Proposition 1.3.8. *Let $\mathcal{E}$ be a quasi-abelian category. The functor*

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

which associates to any object E of $\mathcal{E}$ the complex

$$0 \longrightarrow E \longrightarrow 0$$

where E is in degree 0 is fully faithful.

Remark 1.3.9. Let $X^\cdot$ be an object of $\mathcal{LH}(\mathcal{E})$. By an abuse of notations, we will write

$$X^\cdot \in \mathcal{E}$$

if $X^\cdot$ is isomorphic to $I(E)$ for some object E of $\mathcal{E}$.

Proposition 1.3.10. *Let $\mathcal{E}$ be a quasi-abelian category.*

(a) Any object of $\mathcal{LH}(\mathcal{E})$ is isomorphic to a complex

$$0 \longrightarrow A \xrightarrow{u} B \longrightarrow 0$$

where B is in degree 0 and u is a monomorphism. Moreover, such an object is in the essential image of I if and only if u is strict.

(b) A sequence

$$E \longrightarrow F \longrightarrow G$$

of $\mathcal{E}$ is strictly exact if and only if the sequence

$$I(E) \longrightarrow I(F) \longrightarrow I(G)$$

of $\mathcal{LH}(\mathcal{E})$ is exact.

Corollary 1.3.11. *Let $\mathcal{E}$ be a quasi-abelian category and let $X^\cdot$ be an object of $D(\mathcal{E})$. Then,*

(i) $LH^k(X^\cdot) = 0 \iff X^\cdot$ *is strictly exact in degree k,*

(ii) $LH^k(X^\cdot) \in \mathcal{E} \iff d_X^{k-1}$ *is strict.*

Remark 1.3.12. Replacing the notion of strictly exact sequence by the notion of costrictly exact sequence, we may define a second t-structure on $D(\mathcal{E})$. We call it the *right t-structure* and its associated heart (the *right heart*) is denoted by $\mathcal{RH}(\mathcal{E})$.

1.4 Derivation of functors between quasi-abelian categories

In this section, $F : \mathcal{E} \longrightarrow \mathcal{E}'$ will denote a functor between quasi-abelian categories.

Definition 1.4.1. Let

$$Q : K^+(\mathcal{E}) \longrightarrow D^+(\mathcal{E}) \qquad \text{and} \qquad Q' : K^+(\mathcal{E}') \longrightarrow D^+(\mathcal{E}')$$

be the canonical functors. A *right derived functor of F* is the data of a pair (T, s) where

$$T : D^+(\mathcal{E}) \longrightarrow D^+(\mathcal{E}')$$

is a functor of triangulated categories and

$$s : Q' \circ K^+(F) \longrightarrow T \circ Q$$

is a morphism of functors such that for any pair (T', t) where

$$T' : D^+(\mathcal{E}) \longrightarrow D^+(\mathcal{E}')$$

$$t : Q' \circ K^+(F) \longrightarrow T' \circ Q,$$

there is a unique morphism $\alpha : T \longrightarrow T'$ of functors making the diagram

$$\begin{tikzcd} Q' \circ K^+(F) \arrow[d, "s"'] \arrow[dr, "t"] & \\ T \circ Q \arrow[r, "\alpha \circ \mathrm{id}_Q"'] & T' \circ Q \end{tikzcd}$$

commutative.

Definition 1.4.2. A full *subcategory* $\mathcal{I}$ of $\mathcal{E}$ is F*-injective* if

(i) for any $E \in \mathrm{Ob}(\mathcal{E})$, there is a strict monomorphism $E \longrightarrow I$ where $I \in \mathrm{Ob}(\mathcal{I})$,

(ii) $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is a strictly exact sequence of $\mathcal{E}$ such that E', $E \in \mathrm{Ob}(\mathcal{I})$, then

  (a) $E'' \in \mathrm{Ob}(\mathcal{I})$,

  (b) $0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$ is strictly exact.

Proposition 1.4.3. *If $\mathcal{I}$ is an F-injective subcategory of $\mathcal{E}$, then for any object $X^{\cdot}$ of $C^+(\mathcal{E})$, there is a strict quasi-isomorphism*

$$u^{\cdot} : X^{\cdot} \longrightarrow I^{\cdot}$$

such that, for any k, $I^k \in \mathrm{Ob}(\mathcal{I})$ and $u^k : X^k \longrightarrow I^k$ is a strict monomorphism.

We call $I^{\cdot}$ an F-injective resolution *of $X^{\cdot}$.*

Proposition 1.4.4. *If $\mathcal{E}$ has an F-injective subcategory $\mathcal{I}$, the functor*

$$F : \mathcal{E} \longrightarrow \mathcal{E}'$$

is right derivable and its derived functor

$$RF : D^+(\mathcal{E}) \longrightarrow D^+(\mathcal{E}')$$

is given by

$$RF(X^{\cdot}) = F(I^{\cdot})$$

where $I^{\cdot}$ is an F-injective resolution of $X^{\cdot}$.

Definition 1.4.5. (i) An *object* I of $\mathcal{E}$ is *injective* if for any strict monomorphism $u : E \longrightarrow F$ and any morphism $v : E \longrightarrow I$, there is a morphism $v' : F \longrightarrow I$ making the diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "v"'] & F \arrow[dl, "v'"] \\ I & \end{tikzcd}$$

commutative.

(ii) The category $\mathcal{E}$ has *enough injective objects* if for any object E of $\mathcal{E}$, there is a strict monomorphism $E \longrightarrow I$ with I injective.

Proposition 1.4.6. *If $\mathcal{E}$ has enough injective objects, then the full subcategory $\mathcal{I}$ of $\mathcal{E}$ formed by the injective objects is F-injective for any functor $F : \mathcal{E} \longrightarrow \mathcal{E}'$.*

In particular, any functor $F : \mathcal{E} \longrightarrow \mathcal{E}'$ is right derivable.

Now, let us explain how to derive a bifunctor.

Proposition 1.4.7. *Let*

$$F(\cdot, \cdot) : \mathcal{E} \times \mathcal{E}' \longrightarrow \mathcal{E}''$$

be a bifunctor between quasi-abelian categories. Assume that there are full subcategories $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{E}$ and $\mathcal{E}'$ respectively such that

(i) for any $J \in \mathrm{Ob}(\mathcal{J})$, $\mathcal{I}$ is $F(\cdot, J)$-injective,

(ii) for any $I \in \mathrm{Ob}(\mathcal{I})$, $\mathcal{J}$ is $F(I, \cdot)$-injective.

Then, the functor $F(\cdot, \cdot)$ is right derivable and its derived functor

$$RF : D^+(\mathcal{E}) \times D^+(\mathcal{E}') \longrightarrow D^+(\mathcal{E}'')$$

is given by

$$RF(X^{\cdot}, Y^{\cdot}) = sF(I^{\cdot}, J^{\cdot})$$

where $I^{\cdot}$ (resp. $J^{\cdot}$) is an injective resolution of $X^{\cdot}$ (resp. $Y^{\cdot}$) and $sF(I^{\cdot}, J^{\cdot})$ is the simple complex associated to the double complex $F(I^{\cdot}, J^{\cdot})$.

Remark 1.4.8. Dually, it is possible to derive functors on the left by considering F-projective subcategories.

2 Projective systems in quasi-abelian categories

2.1 Categories of projective systems

Definition 2.1.1. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a small category. We denote by $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$ the category of functors from $\mathcal{I}^{\mathrm{op}}$ to $\mathcal{C}$. The objects of $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$ will be called *projective systems of $\mathcal{C}$ indexed by $\mathcal{I}$.*

Proposition 2.1.2. *Let $\mathcal{I}$ be a small category. Assume $\mathcal{E}$ is a quasi-abelian category. Then, $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ is a quasi-abelian category.*

Proof. We know that the category $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$is additive.

Consider a morphism $f : E \longrightarrow F$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. The kernel of f is given by an object K of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and a morphism $u : K \longrightarrow E$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ such that for any $i \in \mathcal{I}$, the object $K(i)$ of $\mathcal{E}$ and the morphism $u(i) : K(i) \longrightarrow E(i)$ form a kernel of $f(i)$. The cokernel of f is defined similarly. It follows that a morphism $f : E \longrightarrow F$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ is strict if and only if $f(i) : E(i) \longrightarrow F(i)$ is strict in $\mathcal{E}$ for any $i \in \mathcal{I}$.

Consider a cartesian square

$$\begin{tikzcd} E \arrow[r, "f"] & F \\ E' \arrow[u] \arrow[r, "f'"'] & F' \arrow[u] \end{tikzcd}$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, where f is a strict epimorphism. Since for any $i \in \mathcal{I}$, the square

$$\begin{tikzcd} E(i) \arrow[r, "f(i)"] & F(i) \\ E'(i) \arrow[u] \arrow[r, "f'(i)"'] & F'(i) \arrow[u] \end{tikzcd}$$

of $\mathcal{E}$ is cartesian and since $f(i)$ is a strict epimorphism, $f'(i)$ is a strict epimorphism. It follows that f' is a strict epimorphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Using the same kind of arguments, in a cocartesian square

$$\begin{tikzcd} E' \arrow[r, "f'"] & F' \\ E \arrow[u] \arrow[r, "f"'] & F \arrow[u] \end{tikzcd}$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, if f is a strict monomorphism, then f' is also a strict monomorphism. $\square$

2.2 Projective systems of product and coproduct type

Remark 2.2.1. Hereafter, by an abuse of notations, we will denote by the same symbol a set and its associated discrete category.

Definition 2.2.2. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be an additive category with products. We define the functor

$$\Pi : \mathcal{E}^{\mathrm{Ob}(\mathcal{I})} \longrightarrow \mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$$

in the following way. At the level of objects, for any functor $S : \mathrm{Ob}(\mathcal{I}) \longrightarrow \mathcal{E}$ we define the functor

$$\Pi(S) : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

by setting

$$\Pi(S)(i) = \prod_{j \xrightarrow{\alpha} i} S(j)$$

for any $i \in \mathcal{I}$. Let i be an object of $\mathcal{I}$. For any morphism $\alpha : j \longrightarrow i$ of $\mathcal{I}$, we denote by

$$p_{j \xrightarrow{\alpha} i} : \Pi(S)(i) \longrightarrow S(j)$$

the canonical projection. Then, if $f : i' \longrightarrow i$ is a morphism of $\mathcal{I}$, we define

$$\Pi(S)(f) : \Pi(S)(i) \longrightarrow \Pi(S)(i')$$

by setting

$$p_{j' \xrightarrow{\alpha'} i'} \circ \Pi(S)(f) = p_{j' \xrightarrow{f \circ \alpha'} i}$$

for any morphism $\alpha' : j' \longrightarrow i'$.

At the level of morphisms, for any morphism $s : S \longrightarrow S'$ of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, we define

$$\Pi(s) : \Pi(S) \longrightarrow \Pi(S')$$

by setting

$$p_{j \xrightarrow{\alpha} i} \circ \Pi(s)(i) = s(j) \circ p_{j \xrightarrow{\alpha} i}$$

for any object i of $\mathcal{I}$ and any morphism $\alpha : j \longrightarrow i$ of $\mathcal{I}$.

Definition 2.2.3. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be an additive category with coproducts. Applying the preceding definition to $\mathcal{I}^{\mathrm{op}}$ and $\mathcal{E}^{\mathrm{op}}$, we get a functor

$$(\mathcal{E}^{\mathrm{op}})^{\mathrm{Ob}(\mathcal{I}^{\mathrm{op}})} \longrightarrow (\mathcal{E}^{\mathrm{op}})^{(\mathcal{I}^{\mathrm{op}})^{\mathrm{op}}}.$$

Through the canonical isomorphism

$$(\mathcal{C}^{\mathrm{op}})^{(\mathcal{D}^{\mathrm{op}})} \simeq (\mathcal{C}^{\mathcal{D}})^{\mathrm{op}},$$

this gives us a functor

$$\Pi : \mathcal{E}^{\mathrm{Ob}(\mathcal{I})} \longrightarrow \mathcal{E}^{\mathcal{I}^{\mathrm{op}}}.$$

Note that

$$\Pi(S)(i) = \prod_{i \xrightarrow{\alpha} j} S(j).$$

Definition 2.2.4. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be an additive category with products (resp. with coproducts). A *projective system*

$$E : \mathcal{I}^{\mathrm{op}} \to \mathcal{E}$$

is of *product type* (resp. of *coproduct type*) if there is an object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$ such that

$$E \simeq \Pi(S) \qquad (\text{resp. } E \simeq \amalg(S))$$

in $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Definition 2.2.5. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be an arbitrary category. We define the functor

$$\mathrm{O} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$$

by

$$\mathrm{O}(E)(i) = E(i) \qquad \forall i \in \mathcal{I}$$

for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. If $f : E \longrightarrow E'$ is a morphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we define

$$\mathrm{O}(f) : \mathrm{O}(E) \longrightarrow \mathrm{O}(E')$$

by setting

$$\mathrm{O}(f)(i) = f(i).$$

One checks easily that we have:

Proposition 2.2.6. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be an additive category with products (resp. with coproducts). For any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have*

$$\mathrm{Hom}_{\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}}(\mathrm{O}(E), S) \simeq \mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(E, \Pi(S))$$

$$(\textit{resp. } \mathrm{Hom}_{\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}}(S, \mathrm{O}(E)) \simeq \mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(\amalg(S), E)).$$

Proposition 2.2.7. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be an additive category with products (resp. with coproducts). For any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, we have the isomorphism*

$$\varprojlim_{i \in \mathcal{I}} \Pi(S)(i) \simeq \prod_{i \in \mathcal{I}} S(i)$$

$$(\textit{resp. } \varinjlim_{i \in \mathcal{I}} \amalg(S)(i) \simeq \coprod_{i \in \mathcal{I}} S(i)).$$

Proof. This follows directly from the definition of the projective and the inductive limits. □

2.3 Injective and projective objects

Proposition 2.3.1. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. If S is an injective object of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, then $\Pi(S)$ is an injective object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.*

Proof. Let S be an injective object of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$. Consider a strict monomorphism

$$f : E \longrightarrow E'$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. Since for any $i \in \mathcal{I}$, $f(i) : E(i) \longrightarrow E'(i)$ is a strict monomorphism and since $S(i)$ is injective in $\mathcal{E}$, the sequence

$$\mathrm{Hom}_{\mathcal{E}}(E'(i), S(i)) \xrightarrow{\mathrm{Hom}\,(f(i),S(i))} \mathrm{Hom}_{\mathcal{E}}(E(i), S(i)) \longrightarrow 0$$

is exact. It follows that the sequence

$$\mathrm{Hom}_{\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}}(\mathrm{O}(E'), S) \xrightarrow{\mathrm{Hom}\,(\mathrm{O}(f),S)} \mathrm{Hom}_{\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}}(\mathrm{O}(E), S) \longrightarrow 0$$

is exact. By Proposition 2.2.6, the sequence

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(E', \Pi(S)) \xrightarrow{\mathrm{Hom}\,(f,\Pi(S))} \mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(E, \Pi(S)) \longrightarrow 0$$

is also exact and the conclusion follows. □

Proposition 2.3.2. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products (resp. with coproducts). For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is a strict monomorphism (resp. strict epimorphism)*

$$f : E \longrightarrow \Pi(\mathrm{O}(E)) \qquad (\textit{resp. } g : \amalg(\mathrm{O}(E)) \longrightarrow E)$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Proof. Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. We define the morphism

$$f : E \longrightarrow \Pi(\mathrm{O}(E))$$

by setting

$$p_{j \xrightarrow{\alpha} i} \circ f(i) = E(\alpha)$$

for any object i of $\mathcal{I}$ and any morphism $\alpha : j \longrightarrow i$ of $\mathcal{I}$. Since for any object i of $\mathcal{I}$, we have

$$p_{i \xrightarrow{\mathrm{id}} i} \circ f(i) = E(\mathrm{id}) = \mathrm{id},$$

$p_{i \xrightarrow{\mathrm{id}} i} \circ f(i)$ is a strict monomorphism. Consequently, for any $i \in \mathcal{I}$, $f(i)$ is a strict monomorphism of $\mathcal{E}$ and f is a strict monomorphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. □

Proposition 2.3.3. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. If $\mathcal{E}$ has enough injective objects then the category $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ has enough injective objects.*

Proof. Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. We know that there is a strict monomorphism

$$f : E \longrightarrow \Pi(\mathrm{O}(E))$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. Moreover, since $\mathcal{E}$ has enough injective objects, for any $i \in \mathcal{I}$, there is a strict monomorphism

$$s(i) : E(i) \rightarrow I(i)$$

of $\mathcal{E}$, where $I(i)$ is an injective object of $\mathcal{E}$. These morphisms define a morphism

$$s : \mathrm{O}(E) \longrightarrow I$$

of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$ where I is an injective object of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$. Now, consider the morphism

$$\Pi(s) \circ f : E \longrightarrow \Pi(I)$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. Since the product of strict monomorphisms is a strict monomorphism, for any $i \in \mathcal{I}$, $\Pi(s)(i)$ is a strict monomorphism of $\mathcal{E}$. Consequently, $\Pi(s)$ is a strict monomorphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and $\Pi(s) \circ f$ is a strict monomorphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Finally, by Proposition 2.3.1, the object $\Pi(I)$ is injective in $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. And the conclusion follows. □

Dually, we have:

Proposition 2.3.4. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with coproducts. If $\mathcal{E}$ has enough projective objects, then the category $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ has enough projective objects.*

2.4 Index restriction and extension

To fix the notations, let us recall a few definitions of the theory of categories.

Definition 2.4.1. Let $F : \mathcal{A} \longrightarrow \mathcal{C}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be two functors between arbitrary categories. We denote by

$$F \downarrow G$$

the category whose objects are the triples

$$(a, f, b)$$

where a is an object of $\mathcal{A}$, b is an object of $\mathcal{B}$ and $f : F(a) \longrightarrow G(b)$ is a morphism of $\mathcal{C}$. If (a, f, b) and (a', f', b') are two objects of $F \downarrow G$, a morphism

$$(u, v) : (a, f, b) \longrightarrow (a', f', b')$$

of $F \downarrow G$ is the data of a morphism $u : a \longrightarrow a'$ of $\mathcal{A}$ and a morphism $v : b \longrightarrow b'$ of $\mathcal{B}$ such that the diagram

$$\begin{tikzcd} F(a) \arrow[r, "f"] \arrow[d, "F(u)"'] & G(b) \arrow[d, "G(v)"] \\ F(a') \arrow[r, "f'"'] & G(b') \end{tikzcd}$$

is commutative.

Remark 2.4.2. Let 1 denote the category with one object.

(i) If the functor $F : 1 \longrightarrow \mathcal{C}$ associates to the object of 1, the object i of $\mathcal{C}$, the category $F \downarrow G$ will simply be denoted $i \downarrow G$.

If, moreover, $G = \mathrm{id}_{\mathcal{C}}$, then the category $i \downarrow G$ will be denoted $i \downarrow \mathcal{C}$.

(ii) Similarly, if the functor $G : 1 \longrightarrow \mathcal{C}$ associates to the object of 1, the object i of $\mathcal{C}$, the category $F \downarrow G$ will be denoted $F \downarrow i$.

Moreover, if $F = \mathrm{id}_{\mathcal{C}}$, then the category $F \downarrow i$ will be denoted $\mathcal{C} \downarrow i$.

Proposition 2.4.3. *Let $\mathcal{I}$ be an arbitrary category. For any object i of $\mathcal{I}$,*

$$(\mathrm{id}_i, i) \qquad (\textit{resp. } (i, \mathrm{id}_i))$$

is an initial object (resp. a terminal object) of $i \downarrow \mathcal{I}$ (resp. $\mathcal{I} \downarrow i$).

Definition 2.4.4. Let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two arbitrary categories and let i be an object of $\mathcal{I}$.

(i) We define the functor

$$J^i : i \downarrow J \longrightarrow \mathcal{J}$$

by setting

$$J^i(f, j) = j$$

for any object (f, j) of $i \downarrow J$ and by setting

$$J^i(\beta) = \beta$$

for any morphism $\beta : (f, j) \longrightarrow (f', j')$ of $i \downarrow J$.

(ii) We may define the functor

$$J_i : J \downarrow i \longrightarrow \mathcal{J}$$

in the same way.

Proposition 2.4.5. *Let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between arbitrary categories and let $\alpha : i' \longrightarrow i$ be a morphism of $\mathcal{I}$.*

(i) There is a functor

$$J^\alpha : i \downarrow J \longrightarrow i' \downarrow J$$

such that

$$J^{i'} \circ J^\alpha = J^i.$$

(ii) There is a functor

$$J_\alpha : J \downarrow i' \longrightarrow J \downarrow i$$

such that

$$J_i \circ J_\alpha = J_{i'}.$$

Proof. The functor

$$J^\alpha : i \downarrow J \longrightarrow i' \downarrow J$$

defined by

$$J^\alpha(f, j) = (f \circ \alpha, j)$$

for any object (f, j) of $i \downarrow J$ and by

$$J^\alpha(u) = u$$

for any morphism $u : (f_1, j_1) \longrightarrow (f_2, j_2)$ of $i \downarrow J$, solves the problem.

The functor J_α is defined similarly. $\square$

Remark 2.4.6. If $J = \mathrm{id}_{\mathcal{I}}$ then the functors

$$J^i, \quad J_i, \quad J^\alpha \quad \text{and} \quad J_\alpha$$

will be denoted respectively by

$$\mathcal{I}^i : i \downarrow \mathcal{I} \longrightarrow \mathcal{I}, \qquad \mathcal{I}_i : \mathcal{I} \downarrow i \longrightarrow \mathcal{I},$$

$$\mathcal{I}^\alpha : i \downarrow \mathcal{I} \longrightarrow i' \downarrow \mathcal{I} \qquad \text{and} \qquad \mathcal{I}_\alpha : \mathcal{I} \downarrow i' \longrightarrow \mathcal{I} \downarrow i.$$

Definition 2.4.7. Let $\mathcal{C}$ be an arbitrary category and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small categories. We define the functor

$$J^+ : \mathcal{C}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{C}^{\mathcal{J}^{\mathrm{op}}}$$

by setting

$$J^+(C) = C \circ J^{\mathrm{op}}$$

for any object C of $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$. If $f : C \longrightarrow C'$ is a morphism of $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$, we define

$$J^+(f) : J^+(C) \longrightarrow J^+(C')$$

by setting

$$J^+(f)(j) = f(J(j))$$

for any $j \in \mathcal{J}$.

Definition 2.4.8. Let $\mathcal{C}$ be a cocomplete category and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small categories. We define the functor

$$J_+ : \mathcal{C}^{\mathcal{J}^{\mathrm{op}}} \longrightarrow \mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$$

in the following way. At the level of objects, for any functor $G : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$, we define the functor

$$J_+(G) : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

by setting

$$J_+(G)(i) = \varinjlim_{(f,j) \in i \downarrow J} (G \circ J^{i,\mathrm{op}})(f, j) = \varinjlim_{(f,j) \in i \downarrow J} G(j)$$

for any $i \in \mathcal{I}$. Let i be an object of $\mathcal{I}$. For any object (f, j) of $i \downarrow J$, denote by

$$r^i_{(f,j)} : G(j) \longrightarrow J_+(G)(i)$$

the canonical morphism. Then, if $\alpha : i' \longrightarrow i$ is a morphism of $\mathcal{I}$, we define

$$J_+(G)(\alpha) : \varinjlim_{(f,j) \in i \downarrow J} G(j) \longrightarrow \varinjlim_{(f',j') \in i' \downarrow J} G(j')$$

by setting

$$J_+(G)(\alpha) \circ r^i_{(f,j)} = r^{i'}_{(f\circ\alpha,j)}$$

for any object (f, j) of $i \downarrow J$.

At the level of morphisms, for any morphism $g : G \longrightarrow G'$ of $\mathcal{C}^{\mathcal{J}^{\mathrm{op}}}$, we define

$$J_+(g) : J_+(G) \longrightarrow J_+(G')$$

by setting

$$J_+(g)(i) \circ r^i_{(f,j)} = r^i_{(f,j)} \circ g(j)$$

for any object i of $\mathcal{I}$ and any object (f, j) of $i \downarrow J$.

One can check easily that we have:

Proposition 2.4.9. *Let $\mathcal{C}$ be a cocomplete category and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between small categories. For any object C of $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$ and any object G of $\mathcal{C}^{\mathcal{J}^{\mathrm{op}}}$, we have*

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}}(J_+(G), C) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}^{\mathrm{op}}}}(G, J^+(C)).$$

3 Derivation of the projective limit functor

3.1 The case where $\mathcal{E}$ has enough injective objects

Proposition 3.1.1. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. If $\mathcal{E}$ has enough injective objects, then the functor*

$$\varprojlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

is right derivable.

Proof. This follows directly from Proposition 2.3.3. □

Dually, we have:

Proposition 3.1.2. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with coproducts. If $\mathcal{E}$ has enough projective objects, then the functor*

$$\varinjlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}} \longrightarrow \mathcal{E}$$

is left derivable.

3.2 Roos complexes

In this section, $\mathcal{I}$ will denote a small category and $\mathcal{E}$ a quasi-abelian category with products.

Definition 3.2.1. We define the functor

$$R^{\cdot}(\mathcal{I}, \cdot) : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow C^{+}(\mathcal{E})$$

in the following way. At the level of objects, for any functor $E : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$, we define

$$R^{\cdot}(\mathcal{I}, E) \in C^{+}(\mathcal{E})$$

by setting

$$R^{n}(\mathcal{I}, E) = 0 \qquad \forall n < 0$$

and

$$R^{n}(\mathcal{I}, E) = \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} E(i_0) \qquad \forall n \geq 0,$$

where

$$i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n$$

is a chain of morphisms of $\mathcal{I}$. Denoting by

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} : R^{n}(\mathcal{I}, E) \longrightarrow E(i_0)$$

the canonical projection, we define the differential

$$d^{n}_{R^{\cdot}(\mathcal{I},E)} : R^{n}(\mathcal{I}, E) \longrightarrow R^{n+1}(\mathcal{I}, E)$$

by setting

$$\begin{aligned} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \circ d^{n}_{R^{\cdot}(\mathcal{I},E)} &= E(\alpha_1) \circ p_{i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \\ &+ \sum_{l=1}^{n} (-1)^{l} p_{i_0 \xrightarrow{\alpha_1} \cdots i_{l-1} \xrightarrow{\alpha_{l+1} \circ \alpha_l} i_{l+1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \\ &+ (-1)^{n+1} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n}. \end{aligned}$$

At the level of morphisms, for any morphism $f : E \longrightarrow E'$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we define

$$R^{\cdot}(\mathcal{I}, f) : R^{\cdot}(\mathcal{I}, E) \longrightarrow R^{\cdot}(\mathcal{I}, E')$$

by setting

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} \circ R^{n}(\mathcal{I}, f) = f(i_0) \circ p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n}.$$

Notation 3.2.2. Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. For any $i \in \mathcal{I}$, we denote by

$$q_i : \varprojlim_{i \in \mathcal{I}} E(i) \longrightarrow E(i)$$

the canonical morphism.

Proposition 3.2.3. *For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is a canonical isomorphism*

$$\epsilon^0(\mathcal{I}, E) : \varprojlim_{i \in \mathcal{I}} E(i) \xrightarrow{\sim} \ker d^0_{R^{\cdot}(\mathcal{I},E)}$$

which induces a canonical morphism

$$\epsilon^{\cdot}(\mathcal{I}, E) : \varprojlim_{i \in \mathcal{I}} E(i) \longrightarrow R^{\cdot}(\mathcal{I}, E).$$

Proof. We define the morphism

$$\epsilon^0(\mathcal{I}, E) : \varprojlim_{i \in \mathcal{I}} E(i) \longrightarrow R^0(\mathcal{I}, E) = \prod_{i \in \mathcal{I}} E(i)$$

by setting

$$p_i \circ \epsilon^0(\mathcal{I}, E) = q_i$$

for any $i \in \mathcal{I}$. Since $d^0_{R^{\cdot}(\mathcal{I},E)} \circ \epsilon^0(\mathcal{I}, E) = 0$, $\epsilon^0(\mathcal{I}, E)$ induces a morphism

$$\epsilon^{\cdot}(\mathcal{I}, E) : \varprojlim_{i \in \mathcal{I}} E(i) \longrightarrow R^{\cdot}(\mathcal{I}, E)$$

of $C^+(\mathcal{E})$. It follows directly from the definitions that $(\varprojlim_{i \in \mathcal{I}} E(i), \epsilon^0(\mathcal{I}, E))$ is a kernel of $d^0_{R^{\cdot}(\mathcal{I},E)}$. □

Definition 3.2.4. Let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between small categories. We define the morphism of functors

$$R^{\cdot}(J, \cdot) : R^{\cdot}(\mathcal{I}, \cdot) \longrightarrow R^{\cdot}(\mathcal{J}, J^+(\cdot))$$

by setting

$$p_{j_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} j_n} \circ R^n(J, E) = p_{J(j_0) \xrightarrow{J(\beta_1)} \cdots \xrightarrow{J(\beta_n)} J(j_n)} \qquad \forall n \geq 0$$

for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Definition 3.2.5. We define the functor

$$R^{\cdot}(\cdot) : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow C^{+}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$$

in the following way. At the level of objects, for any functor $E : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$, we define

$$R^{\cdot}(E) \in C^{+}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$$

by setting

$$R^{\cdot}(E)(i) = R^{\cdot}(\mathcal{I} \downarrow i, \mathcal{I}_i^{+}(E))$$

for any $i \in \mathcal{I}$. If $\alpha : i' \longrightarrow i$ is a morphism of $\mathcal{I}$, we define

$$R^{\cdot}(E)(\alpha) : R^{\cdot}(\mathcal{I} \downarrow i, \mathcal{I}_i^{+}(E)) \longrightarrow R^{\cdot}(\mathcal{I} \downarrow i', \mathcal{I}_{i'}^{+}(E))$$

by setting

$$R^{\cdot}(E)(\alpha) = R^{\cdot}(\mathcal{I}_\alpha, \mathcal{I}_i^{+}(E))$$

and using the fact that $\mathcal{I}_\alpha^{+} \circ \mathcal{I}_i^{+} = (\mathcal{I}_i \circ \mathcal{I}_\alpha)^{+} = \mathcal{I}_{i'}^{+}$. For any $n \geq 0$, we define the differential

$$d^n_{R^{\cdot}(E)} : R^n(E) \longrightarrow R^{n+1}(E)$$

by setting

$$d^n_{R^{\cdot}(E)}(i) = d^n_{R^{\cdot}(\mathcal{I} \downarrow i, \mathcal{I}_i^{+}(E))}$$

for any object i of $\mathcal{I}$.

At the level of morphisms, if $f : E \longrightarrow E'$ is a morphism of functors, we define

$$R^{\cdot}(f) : R^{\cdot}(E) \longrightarrow R^{\cdot}(E')$$

by setting

$$R^{\cdot}(f)(i) = R^{\cdot}(\mathcal{I} \downarrow i, \mathcal{I}_i^{+}(f))$$

for any object i of $\mathcal{I}$.

Remark 3.2.6. Let us notice that to give a chain of morphisms

$$(i_0, f_0) \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} (i_n, f_n)$$

of $\mathcal{I} \downarrow i$ is equivalent to give a chain of morphisms of $\mathcal{I}$ of the form

$$i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{f} i.$$

It follows that for any $n \geq 0$ and any $i \in \mathcal{I}$

$$R^n(E)(i) = \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{f} i} E(i_0)$$

and that for any morphism $\alpha : i' \to i$ of $\mathcal{I}$

$$R^n(E)(\alpha) : \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{f} i} E(i_0) \to \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{f'} i'} E(i_0)$$

is defined by

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{f'} i'} \circ R^n(E)(\alpha) = p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{\alpha \circ f'} i}.$$

For any $n \geq 0$, the differential

$$d^n_{R^\cdot(E)} : R^n(E) \longrightarrow R^{n+1}(E)$$

is given by

$$\begin{aligned} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1} \xrightarrow{f} i} \circ d^n_{R^\cdot(E)}(i) = E(\alpha_1) \circ p_{i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1} \xrightarrow{f} i} \\ + \sum_{l=1}^{n} (-1)^l p_{i_0 \xrightarrow{\alpha_1} \cdots i_{l-1} \xrightarrow{\alpha_{l+1} \circ \alpha_l} i_{l+1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1} \xrightarrow{f} i} \\ + (-1)^{n+1} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{f \circ \alpha_{n+1}} i} \end{aligned}$$

for any $i \in \mathcal{I}$. Finally, for any morphism $f : E \longrightarrow E'$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$

$$R^\cdot(f) : R^\cdot(E) \longrightarrow R^\cdot(E')$$

is given by

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{g} i} \circ R^n(f)(i) = f(i_0) \circ p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n \xrightarrow{g} i}$$

for any $n \geq 0$ and any $i \in \mathcal{I}$.

Lemma 3.2.7. *If the category $\mathcal{I}$ has a terminal object i_∞, then for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is a canonical homotopy equivalence*

$$E(i_\infty) \to R^\cdot(\mathcal{I}, E)$$

Proof. For any $n \geq 0$, define

$$h^n : R^n(\mathcal{I}, E) \longrightarrow R^{n-1}(\mathcal{I}, E)$$

by setting $h^0 = 0$ and

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} i_{n-1}} \circ h^n = (-1)^n p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} i_{n-1} \xrightarrow{\alpha_{i_\infty, i_{n-1}}} i_\infty} \qquad n \geq 1$$

where $\alpha_{i_\infty,i_{n-1}}$ is the unique morphism of $\operatorname{Hom}_{\mathcal{I}}(i_{n-1}, i_\infty)$. Define the canonical morphism

$$f^0 : E(i_\infty) \longrightarrow R^0(\mathcal{I}, E)$$

by setting

$$p_i \circ f^0 = E(\alpha_{i_\infty,i}).$$

Since $d^0_{R^\cdot(\mathcal{I},E)} \circ f^0 = 0$, f^0 induces a canonical morphism

$$f^\cdot : E(i_\infty) \longrightarrow R^\cdot(\mathcal{I}, E).$$

We will also consider the morphism

$$g^\cdot : R^\cdot(\mathcal{I}, E) \longrightarrow E(i_\infty)$$

defined by setting

$$g^k = \begin{cases} p_{i_\infty} & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \geq 1. \end{cases}$$

Clearly,

$$g^\cdot \circ f^\cdot = \operatorname{id}.$$

Moreover, $f^\cdot \circ g^\cdot$ is homotopic to the identity map since we have

$$h^1 \circ d^0_{R^\cdot(\mathcal{I},E)} = \operatorname{id} - f^0 \circ g^0$$

and

$$d^{n-1}_{R^\cdot(\mathcal{I},E)} \circ h^n + h^{n+1} \circ d^n_{R^\cdot(\mathcal{I},E)} = \operatorname{id}_{R^n(\mathcal{I},E)}.$$

□

Proposition 3.2.8. *For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is an isomorphism*

$$\epsilon^\cdot(E) : E \xrightarrow{\sim} R^\cdot(E)$$

in $D^+(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$.

Proof. Let i be an object of $\mathcal{I}$. Since (i, id_i) is a terminal object of $\mathcal{I} \downarrow i$, by Lemma 3.2.7

$$f^\cdot(i) : \mathcal{I}_i^+(E)(i, \operatorname{id}_i) = E(i) \xrightarrow{\sim} R^\cdot(\mathcal{I} \downarrow i, \mathcal{I}_i^+(E)) = R^\cdot(E)(i)$$

in $K^+(\mathcal{E})$, where

$$f^0(i) : E(i) \longrightarrow R^0(E)(i) = \prod_{i_0 \xrightarrow{\alpha} i} E(i_0)$$

is defined by

$$p_{i_0 \xrightarrow{\alpha} i} \circ f^0(i) = E(\alpha).$$

So, for any $i \in \mathcal{I}$, we define

$$\epsilon^0(E)(i) : E(i) \longrightarrow R^0(E)(i)$$

by setting

$$\epsilon^0(E)(i) = f^0(i).$$

Since $d^0_{R^{\cdot}(E)} \circ \epsilon^0(E) = 0$, $\epsilon^0(E)$ induces a morphism

$$\epsilon^{\cdot}(E) : E \longrightarrow R^{\cdot}(E)$$

of $C^+(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$. By construction, for any $i \in \mathcal{I}$, we have the isomorphism

$$\epsilon^{\cdot}(E)(i) : E(i) \xrightarrow{\sim} R^{\cdot}(E)(i)$$

in $D^+(\mathcal{E})$. It follows that

$$\epsilon^{\cdot}(E) : E \xrightarrow{\sim} R^{\cdot}(E)$$

in $D^+(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$. □

Lemma 3.2.9. *For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and any $n \geq 0$, there is an object $S^n(E)$ of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$ such that*

$$R^n(E) \simeq \Pi(S^n(E)).$$

Proof. One checks easily that the functor

$$S^n(E) : \mathrm{Ob}(\mathcal{I}) \longrightarrow \mathcal{E}$$

defined by

$$S^n(E)(i_n) = \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} E(i_0)$$

for any $i_n \in \mathcal{I}$ solves the problem. □

Proposition 3.2.10. *For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is an isomorphism*

$$\varprojlim_{i \in \mathcal{I}} R^{\cdot}(E)(i) \xrightarrow{\sim} R^{\cdot}(\mathcal{I}, E)$$

of $C^+(\mathcal{E})$.

Proof. This follows from the chain of isomorphisms

$$\begin{aligned}
\varprojlim_{i\in\mathcal{I}} R^n(E)(i) &\simeq \varprojlim_{i\in\mathcal{I}} \Pi(S^n(E))(i)\\
&\simeq \prod_{i_n\in\mathcal{I}} S^n(E)(i_n)\\
&\simeq \prod_{i_n\in\mathcal{I}} \prod_{i_0\xrightarrow{\alpha_1}\cdots\xrightarrow{\alpha_n}i_n} E(i_0)\\
&\simeq \prod_{i_0\xrightarrow{\alpha_1}\cdots\xrightarrow{\alpha_n}i_n} E(i_0)\\
&\simeq R^n(\mathcal{I},E).
\end{aligned}$$

□

3.3 The case where $\mathcal{E}$ has exact products

In this section, $\mathcal{I}$ will denote a small category and $\mathcal{E}$ a quasi-abelian category with products.

Definition 3.3.1. An object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ is a *Roos-acyclic projective system* if the co-augmented complex

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow R^0(\mathcal{I},E) \longrightarrow R^1(\mathcal{I},E) \longrightarrow \cdots$$

is strictly exact. In other words, E is Roos-acyclic if and only if

$$LH^k(R^\cdot(\mathcal{I},E)) = 0$$

for any $k > 0$.

Proposition 3.3.2. *For any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, there is a canonical homotopy equivalence*

$$\prod_{j\in\mathcal{I}} S(j) \longrightarrow R^\cdot(\mathcal{I},\Pi(S)).$$

In particular, $\Pi(S)$ is a Roos-acyclic projective system.

Proof. For any $n \geq 0$, define

$$h^n : R^n(\mathcal{I},\Pi(S)) \longrightarrow R^{n-1}(\mathcal{I},\Pi(S))$$

by setting $h^0 = 0$ and

$$p_{j\xrightarrow{\beta}i_0} \circ p_{i_0\xrightarrow{\alpha_1}\cdots\xrightarrow{\alpha_{n-1}}i_{n-1}} \circ h^n = p_{j\xrightarrow{\mathrm{id}}j} \circ p_{j\xrightarrow{\beta}i_0\xrightarrow{\alpha_1}\cdots\xrightarrow{\alpha_{n-1}}i_{n-1}}$$

for $n \geq 1$. Define the canonical morphism

$$u^0 : \prod_{j \in \mathcal{I}} S(j) \to R^0(\mathcal{I}, \Pi(S))$$

by setting

$$p_{j \xrightarrow{\beta} i_0} \circ p_{i_0} \circ u^0 = p_j .$$

Since $d^0_{R^{\cdot}(\mathcal{I},\Pi(S))} \circ u^0 = 0$, u^0 induces a canonical morphism

$$u^{\cdot} : \prod_{j \in \mathcal{I}} S(j) \to R^{\cdot}(\mathcal{I}, \Pi(S)).$$

We will also consider the morphism

$$v^0 : R^0(\mathcal{I}, \Pi(S)) \to \prod_{j \in \mathcal{I}} S(j)$$

defined by setting

$$p_j \circ v^0 = p_{j \xrightarrow{\mathrm{id}} j} \circ p_j$$

and the induced morphism

$$v^{\cdot} : R^{\cdot}(\mathcal{I}, \Pi(S)) \to \prod_{j \in \mathcal{I}} S(j).$$

Clearly,

$$v^{\cdot} \circ u^{\cdot} = \mathrm{id} .$$

Moreover, $u^{\cdot} \circ v^{\cdot}$ is homotopic to the identity map since we have

$$h^1 \circ d^0_{R^{\cdot}(\mathcal{I},\Pi(S))} = \mathrm{id} - u^0 \circ v^0$$

and

$$d^{n-1}_{R^{\cdot}(\mathcal{I},\Pi(S))} \circ h^n + h^{n+1} \circ d^n_{R^{\cdot}(\mathcal{I},\Pi(S))} = \mathrm{id}_{R^n(\mathcal{I},\Pi(S))}$$

for $n \geq 1$. □

Proposition 3.3.3. *Assume $\mathcal{E}$ has exact products. Then, the functor*

$$\varprojlim_{i \in \mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{E}$$

is right derivable.

Proof. It is sufficient to show that the family

$$\mathcal{F} = \{E \in \mathrm{Ob}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}) : E \text{ is Roos-acyclic}\}$$

is $\varprojlim_{i\in\mathcal{I}}$-injective.

(i) Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. By Proposition 2.3.2, there is a strict monomorphism

$$E \longrightarrow \Pi(\mathrm{O}(E))$$

and by the preceding proposition, $\Pi(\mathrm{O}(E))$ belongs to $\mathcal{F}$.

(ii) Consider a strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ where E' and E belong to $\mathcal{F}$. Since $\mathcal{E}$ has exact products, the sequence

$$0 \longrightarrow R^{\cdot}(\mathcal{I}, E') \longrightarrow R^{\cdot}(\mathcal{I}, E) \longrightarrow R^{\cdot}(\mathcal{I}, E'') \longrightarrow 0$$

is strictly exact and gives rise to the long exact sequence

$$\begin{array}{l} 0 \longrightarrow LH^0(R^{\cdot}(\mathcal{I}, E')) \longrightarrow LH^0(R^{\cdot}(\mathcal{I}, E)) \longrightarrow LH^0(R^{\cdot}(\mathcal{I}, E'')) \\ \longrightarrow LH^1(R^{\cdot}(\mathcal{I}, E')) \longrightarrow LH^1(R^{\cdot}(\mathcal{I}, E)) \longrightarrow LH^1(R^{\cdot}(\mathcal{I}, E'')) \\ \longrightarrow LH^2(R^{\cdot}(\mathcal{I}, E')) \longrightarrow LH^2(R^{\cdot}(\mathcal{I}, E)) \longrightarrow LH^2(R^{\cdot}(\mathcal{I}, E'')) \cdots \end{array}$$

Since E' and E are objects of $\mathcal{F}$,

$$LH^k(R^{\cdot}(\mathcal{I}, E')) = LH^k(R^{\cdot}(\mathcal{I}, E)) = 0 \qquad \forall k > 0.$$

It follows that

$$LH^k(R^{\cdot}(\mathcal{I}, E'')) = 0 \qquad \forall k > 0$$

and that E'' belongs to $\mathcal{F}$.

Moreover, by Proposition 3.2.3, for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$,

$$LH^0(R^{\cdot}(\mathcal{I}, E)) \simeq \varprojlim_{i\in\mathcal{I}} E(i)$$

and the preceding long exact sequence shows that the sequence

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} E'(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} E''(i) \longrightarrow 0$$

is exact. $\square$

Proposition 3.3.4. *Assume $\mathcal{E}$ has exact products. Then, for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have a canonical isomorphism*

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq R^{\cdot}(\mathcal{I}, E).$$

Proof. By Proposition 3.2.8 and Lemma 3.2.9, $R^{\cdot}(E)$ is a Roos-acyclic resolution of E. Then,

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \varprojlim_{i\in\mathcal{I}} R^{\cdot}(E)(i) \simeq R^{\cdot}(\mathcal{I}, E)$$

where the last isomorphism follows from Proposition 3.2.10. □

3.4 Derived projective limit functor and index restriction

In this section, $\mathcal{E}$ will denote a quasi-abelian category with products and $J : \mathcal{J} \longrightarrow \mathcal{I}$ a functor between small categories.

Definition 3.4.1. We define the morphism of functors

$$\rho^{\cdot}(J, .) : J^{+}(R^{\cdot}(\cdot)) \longrightarrow R^{\cdot}(J^{+}(\cdot))$$

by setting

$$p_{j_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} j_n \xrightarrow{g} j} \circ \rho^{n}(J, E)(j) = p_{J(j_0) \xrightarrow{J(\beta_1)} \cdots \xrightarrow{J(\beta_n)} J(j_n) \xrightarrow{J(g)} J(j)}$$

for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, any $j \in \mathcal{J}$ and any $n \geq 0$.

Proposition 3.4.2. *The canonical morphism of functors*

$$q_J(\cdot) : \varprojlim_{i\in\mathcal{I}} \longrightarrow \varprojlim_{j\in\mathcal{J}} \circ J^{+}$$

characterized by the fact that

$$q_j \circ q_J(E) = q_{J(j)}$$

for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and any $j \in \mathcal{J}$ induces a canonical morphism

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} \circ J^{+}.$$

Moreover, if $\mathcal{E}$ has exact products, the diagram

$$\begin{tikzcd}
\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \arrow[r, "\sim"] \arrow[d] & R^{\cdot}(\mathcal{I}, E) \arrow[d, "R^{\cdot}(J,E)"] \\
\mathrm{R}\varprojlim_{j\in\mathcal{J}} J^{+}(E)(j) \arrow[r, "\sim"] & R^{\cdot}(\mathcal{J}, J^{+}(E))
\end{tikzcd}$$

is commutative for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Proof. By a well-known procedure of homological algebra, the canonical morphism

$$q_J(\cdot) : \varprojlim_{i\in\mathcal{I}} \longrightarrow \varprojlim_{j\in\mathcal{J}} \circ J^+$$

induces a canonical morphism

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} \circ RJ^+.$$

Since the functor J^+ is exact, we get the canonical morphism

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} \circ J^+.$$

Assuming $\mathcal{E}$ has exact products, we may visualize the construction of this morphism in the following way. Consider an object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. We know that

$$\epsilon^{\cdot}(E) : E \xrightarrow{\sim} R^{\cdot}(E)$$

is a $\varprojlim_{i\in\mathcal{I}}$-acyclic resolution of E. The functor J^+ being exact, we have the isomorphism

$$J^+(\epsilon^{\cdot}(E)) : J^+(E) \xrightarrow{\sim} J^+(R^{\cdot}(E))$$

in $D^+(\mathcal{E}^{\mathcal{J}^{\mathrm{op}}})$. Since

$$\epsilon^{\cdot}(J^+(E)) : J^+(E) \xrightarrow{\sim} R^{\cdot}(J^+(E))$$

is a $\varprojlim_{j\in\mathcal{J}}$-acyclic resolution of $J^+(E)$,

$$\epsilon^{\cdot}(J^+(E)) \circ (J^+(\epsilon^{\cdot}(E)))^{-1} : J^+(R^{\cdot}(E)) \xrightarrow{\sim} R^{\cdot}(J^+(E))$$

is a $\varprojlim_{j\in\mathcal{J}}$-acyclic resolution of $J^+(R^{\cdot}(E))$. Moreover, the diagram

$$\begin{tikzcd}
J^+(R^{\cdot}(E)) \arrow[r, "\rho^{\cdot}(J,E)", "\sim"'] & R^{\cdot}(J^+(E)) \\
J^+(E) \arrow[u, "J^+(\epsilon^{\cdot}(E))", "\wr"'] \arrow[ur, "\sim", "\epsilon^{\cdot}(J^+(E))"'] &
\end{tikzcd}$$

of $D^+(\mathcal{E}^{\mathcal{J}^{\mathrm{op}}})$ being commutative, we have

$$\epsilon^{\cdot}(J^+(E)) \circ (J^+(\epsilon^{\cdot}(E)))^{-1} = \rho^{\cdot}(J, E).$$

Hence, the canonical morphism

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} J^+(E)(j)$$

is given by the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\varprojlim\limits_{i\in\mathcal{I}} E(i) \arrow[rr, dashed] \arrow[d, "\wr"] & & \mathrm{R}\varprojlim\limits_{j\in\mathcal{J}} J^{+}(E)(j) \arrow[d, "\wr"] \\
\varprojlim\limits_{i\in\mathcal{I}} R^{\cdot}(E)(i) \arrow[r, "q_J(R^{\cdot}(E))"'] & \varprojlim\limits_{j\in\mathcal{J}} J^{+}(R^{\cdot}(E))(j) \arrow[r, "\varprojlim\limits_{j\in\mathcal{J}} \rho^{\cdot}(J,E)(j)"'] & \varprojlim\limits_{j\in\mathcal{J}} R^{\cdot}(J^{+}(E))(j)
\end{tikzcd}$$

Since a direct computation shows that the diagram

$$\begin{tikzcd}
\varprojlim\limits_{i\in\mathcal{I}} R^{\cdot}(E)(i) \arrow[r, "q_J(R^{\cdot}(E))"] \arrow[d, "\wr"] & \varprojlim\limits_{j\in\mathcal{J}} J^{+}(R^{\cdot}(E))(j) \arrow[r, "\varprojlim\limits_{j\in\mathcal{J}} \rho^{\cdot}(J,E)(j)"] & \varprojlim\limits_{j\in\mathcal{J}} R^{\cdot}(J^{+}(E))(j) \arrow[d, "\wr"] \\
R^{\cdot}(\mathcal{I},E) \arrow[rr, "R^{\cdot}(J,E)"'] & & R^{\cdot}(\mathcal{J},J^{+}(E))
\end{tikzcd}$$

is commutative, the conclusion follows. □

3.5 Dual results for the inductive limit functor

In this section $\mathcal{I}$ will denote a small category and $\mathcal{E}$ a quasi-abelian category with coproducts.

By duality, the results and constructions in the preceding sections can be easily adapted to derive the functor

$$\varinjlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}} \to \mathcal{E}.$$

We will not do this explicitly here. However, in the rest of this paper, we will need to work with the derived functor of

$$\varinjlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{E}.$$

To avoid confusions, we will fix below the notations used in this case.

The functor

$$R_{\cdot}(\mathcal{I},\cdot) : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to C^{-}(\mathcal{E})$$

is defined by

$$R_n(\mathcal{I},E) = (R^n(\mathcal{I}^{\mathrm{op}},E^{\mathrm{op}}))^{\mathrm{op}}$$

and the differential is given by

$$d_n^{R_{\cdot}(\mathcal{I},E)} = (d^{n-1}_{R^{\cdot}(\mathcal{I}^{\mathrm{op}},E^{\mathrm{op}})})^{\mathrm{op}}.$$

If $f : E \longrightarrow E'$ is a morphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, then

$$R_n(\mathcal{I}, f) = (R^n(\mathcal{I}^{\mathrm{op}}, f^{\mathrm{op}}))^{\mathrm{op}}.$$

As in Proposition 3.2.3, there is a canonical isomorphism

$$\epsilon_0(\mathcal{I}, E) : \operatorname{coker} d_0^{R_\cdot(\mathcal{I},E)} \xrightarrow{\sim} \varinjlim_{i \in \mathcal{I}} E(i).$$

The functor

$$R_\cdot(\cdot) : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow C^-(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$$

is defined by

$$R_\cdot(E) = (R^\cdot(E^{\mathrm{op}}))^{\mathrm{op}}$$

and the differential is given by

$$d_n^{R_\cdot(E)} = (d^{n-1}_{R^\cdot(E^{\mathrm{op}})})^{\mathrm{op}}.$$

If $f : E \longrightarrow E'$ is a morphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, then

$$R_\cdot(f) = (R^\cdot(f^{\mathrm{op}}))^{\mathrm{op}}.$$

As in Proposition 3.2.8, there is a canonical isomorphism

$$\epsilon_\cdot(E) : R_\cdot(E) \xrightarrow{\sim} E$$

in $D^-(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$. For any $n \geq 0$, there is an object $S_n(E)$ of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$ such that

$$R_n(E) \simeq \amalg(S_n(E)).$$

Moreover, there is a canonical isomorphism

$$R_\cdot(\mathcal{I}, E) \xrightarrow{\sim} \varinjlim_{i \in \mathcal{I}} R_\cdot(E)(i)$$

in $C^-(\mathcal{E})$.

Therefore, as in Propositions 3.3.3 and 3.3.4, if $\mathcal{E}$ has exact coproducts, the functor

$$\varinjlim_{i \in \mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

is left derivable and we have

$$\mathrm{L}\varinjlim_{i \in \mathcal{I}} E(i) \simeq R_\cdot(\mathcal{I}, E).$$

If $J : \mathcal{J} \longrightarrow \mathcal{I}$ is a functor between small categories, then we define the morphism of functors

$$\rho.(J, \cdot) : R.(J^+(\cdot)) \longrightarrow J^+(R.(\cdot))$$

by setting

$$\rho.(J, E) = (\rho^{\cdot}(J^{\mathrm{op}}, E^{\mathrm{op}}))^{\mathrm{op}}.$$

For any $n \geq 0$, we define the morphism

$$\tilde{\rho}_n(J, E) : J_+(R_n(J^+(E))) \longrightarrow R_n(E)$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ as the image of $\rho_n(J, E)$ by the adjunction

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{J}^{\mathrm{op}}}}(R_n(J^+(E)), J^+(R_n(E))) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(J_+(R_n(J^+(E))), R_n(E))$$

The morphisms $\tilde{\rho}_n(J, E)$ induce a morphism of functors

$$\tilde{\rho}.(J, \cdot) : J_+(R.(J^+(\cdot))) \longrightarrow R.(\cdot).$$

3.6 Relations between RHom and derived limits

Definition 3.6.1. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category. Consider an object X of $\mathcal{E}$. We define the functor

$$\mathrm{Hom}\,(X, \cdot) : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$$

in the following way: at the level of objects, if Y is an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we set

$$\mathrm{Hom}\,(X, Y)(i) = \mathrm{Hom}_{\mathcal{E}}(X, Y(i))$$

for any object i of $\mathcal{I}$. If $\alpha : j \longrightarrow i$ is a morphism of $\mathcal{I}$,

$$\mathrm{Hom}\,(X, Y)(\alpha) : \mathrm{Hom}_{\mathcal{E}}(X, Y(i)) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, Y(j))$$

is defined by

$$\mathrm{Hom}\,(X, Y)(\alpha)(f) = \mathrm{Hom}_{\mathcal{E}}(X, Y(\alpha))(f) = Y(\alpha) \circ f$$

for any $f \in \mathrm{Hom}_{\mathcal{E}}(X, Y(i))$.

At the level of morphisms, if $F : Y \longrightarrow Y'$ is a morphism of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$,

$$\mathrm{Hom}\,(X, F) : \mathrm{Hom}\,(X, Y) \longrightarrow \mathrm{Hom}\,(X, Y')$$

is defined by

$$\mathrm{Hom}\,(X, F)(i) = \mathrm{Hom}_{\mathcal{E}}(X, F(i))$$

for any $i \in \mathcal{I}$.

Dually, we define also the functor

$$\mathrm{Hom}\,(\cdot, X) : (\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})^{\mathrm{op}} \longrightarrow \mathcal{A}b^{\mathcal{I}}.$$

Lemma 3.6.2. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category.*

(i) For any object X of $\mathcal{E}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$\mathrm{Hom}\,(X, R^{\cdot}(\mathcal{I}, E)) \simeq R^{\cdot}(\mathcal{I}, \mathrm{Hom}\,(X, E)).$$

(ii) For any object X of $\mathcal{E}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$\mathrm{Hom}\,(R_{\cdot}(\mathcal{I}, E), X) \simeq R^{\cdot}(\mathcal{I}^{\mathrm{op}}, \mathrm{Hom}\,(E, X)).$$

Proof. This follows directly from the definitions. □

Proposition 3.6.3. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with enough injective objects. For any object X of $\mathcal{E}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have*

$$\mathrm{RHom}\,(\mathrm{L}\varinjlim_{i\in\mathcal{I}} E(i), X) \simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}}(\mathrm{RHom}\,(E, X))(i)$$

and

$$\mathrm{RHom}\,(X, \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i)) \simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}}(\mathrm{RHom}\,(X, E))(i).$$

Proof. First, recall that since $\mathcal{E}$ has enough injective objects, coproducts are exact. Hence, the inductive limit functor is left derivable. Let

$$0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be an injective resolution of X. On one hand, $\mathrm{RHom}\,(\mathrm{L}\varinjlim_{i\in\mathcal{I}} E(i), X)$ is given by the simple complex associated to

$$0 \longrightarrow \mathrm{Hom}\,(R_{\cdot}(\mathcal{I}, E), I^0) \longrightarrow \mathrm{Hom}\,(R_{\cdot}(\mathcal{I}, E), I^1) \longrightarrow \cdots$$

This complex is isomorphic to the simple complex associated to

$$0 \longrightarrow R^{\cdot}(\mathcal{I}^{\mathrm{op}}, \mathrm{Hom}\,(E, I^0)) \longrightarrow R^{\cdot}(\mathcal{I}^{\mathrm{op}}, \mathrm{Hom}\,(E, I^1)) \longrightarrow \cdots$$

On the other hand, $\mathrm{RHom}\,(E, X)$ is given by the complex

$$0 \longrightarrow \mathrm{Hom}\,(E, I^0) \longrightarrow \mathrm{Hom}\,(E, I^1) \longrightarrow \cdots$$

Therefore, $\mathrm{R}\varprojlim_{i\in\mathcal{I}}(\mathrm{RHom}\,(E, X))(i)$ is isomorphic to the simple complex associated to

$$0 \longrightarrow \mathrm{R}\varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(E, I^0)(i) \longrightarrow \mathrm{R}\varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(E, I^1)(i) \longrightarrow \cdots$$

Since for any $l \geq 0$

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(E, I^l)(i) \simeq R^{\cdot}(\mathcal{I}^{\mathrm{op}}, \mathrm{Hom}\,(E, I^l)),$$

the first isomorphism is established.

Next, we know that E has an injective resolution of the form

$$0 \longrightarrow \Pi(S^0) \longrightarrow \Pi(S^1) \longrightarrow \cdots$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ such that for $l \geq 0$, S^l is an injective object of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$. On one hand, since

$$\mathrm{Hom}\,(X, \Pi(S^l)) \simeq \Pi\,\mathrm{Hom}\,(X, S^l)$$

and since projective systems of product type are $\varprojlim_{i\in\mathcal{I}}$-acyclic, by composition of the derived functors, we have

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}}(\mathrm{RHom}\,(X, E))(i) \simeq R(\varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X, E)(i)).$$

Hence, $\mathrm{R}\varprojlim_{i\in\mathcal{I}}(\mathrm{RHom}\,(X, E))(i)$ is isomorphic to the complex

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X, \Pi(S^0))(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X, \Pi(S^1))(i) \longrightarrow \cdots$$

Moreover, for any $l \geq 0$, we get

$$\begin{aligned}\varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X, \Pi(S^l))(i) &\simeq \mathrm{Hom}_{\mathcal{E}}(X, \varprojlim_{i\in\mathcal{I}} \Pi(S^l)(i)) \\ &\simeq \mathrm{Hom}_{\mathcal{E}}(X, \prod_{i\in\mathcal{I}} S^l(i)).\end{aligned}$$

Hence, $\mathrm{R}\varprojlim_{i\in\mathcal{I}}(\mathrm{RHom}\,(X, E))(i)$ is given by the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, \prod_{i\in\mathcal{I}} S^0(i)) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, \prod_{i\in\mathcal{I}} S^1(i)) \longrightarrow \cdots$$

On the other hand, $\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i)$ is given by the complex

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} \Pi(S^0)(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} \Pi(S^1)(i) \longrightarrow \cdots$$

This complex is isomorphic to the complex

$$0 \longrightarrow \prod_{i\in\mathcal{I}} S^0(i) \longrightarrow \prod_{i\in\mathcal{I}} S^1(i) \longrightarrow \cdots$$

Since the product of injective objects is an injective object, the last complex is an injective resolution of $\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i)$. Therefore, $\mathrm{RHom}\,(X, \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i))$ is isomorphic to the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, \prod_{i\in\mathcal{I}} S^0(i)) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, \prod_{i\in\mathcal{I}} S^1(i)) \longrightarrow \cdots$$

and the conclusion follows. □

Proposition 3.6.4. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with exact products. Consider an object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. If for any object X of $\mathcal{E}$, $\mathrm{Hom}\,(X, E)$ is $\varprojlim_{i\in\mathcal{I}}$-acyclic, then E is $\varprojlim_{i\in\mathcal{I}}$-acyclic.*

Proof. Consider $X \in \mathrm{Ob}(\mathcal{E})$. We know that

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X, E)(i) \simeq R^{\cdot}(\mathcal{I}, \mathrm{Hom}\,(X, E)) \simeq \mathrm{Hom}\,(X, R^{\cdot}(\mathcal{I}, E)).$$

Since $\mathrm{Hom}\,(X, E)$ is $\varprojlim_{i\in\mathcal{I}}$-acyclic, we have

$$\begin{aligned} \mathrm{Hom}\,(X, R^{\cdot}(\mathcal{I}, E)) &\simeq \varprojlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X, E)(i) \\ &\simeq \mathrm{Hom}\,(X, \varprojlim_{i\in\mathcal{I}} E(i)). \end{aligned}$$

Therefore, the complex $\mathrm{Hom}\,(X, R^{\cdot}(\mathcal{I}, E))$ is exact in degree $k \neq 0$ for any $X \in \mathrm{Ob}(\mathcal{E})$. Hence, Remark 1.2.2 shows that $R^{\cdot}(\mathcal{I}, E)$ is strictly exact in degree $k \neq 0$. It follows that

$$R^{\cdot}(\mathcal{I}, E) \simeq \varprojlim_{i\in\mathcal{I}} E(i)$$

in $D^+(\mathcal{E})$. Since $\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq R^{\cdot}(\mathcal{I}, E)$, we get

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \varprojlim_{i\in\mathcal{I}} E(i).$$

□

4 Derived limits and the symbolic-Hom functor

4.1 The symbolic-Hom functor

In this section, $\mathcal{I}$ will denote a small category and $\mathcal{E}$ a complete additive category.

Definition 4.1.1. We denote by

$$[\cdot,\cdot]_{\mathcal{I}} : (\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})^{\mathrm{op}} \times \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

the *symbolic*-Hom *functor*. For any object M of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, the object $[M,E]_{\mathcal{I}}$ of $\mathcal{E}$ is characterized by

$$\mathrm{Hom}_{\mathcal{E}}(X,[M,E]_{\mathcal{I}}) \simeq \mathrm{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(M, h_X \circ E) \qquad \forall X \in \mathrm{Ob}(\mathcal{E}).$$

If $\mathcal{I}$ is the one point category, then $[\cdot,\cdot]_{\mathcal{I}}$ will be denoted by

$$[\cdot,\cdot] : \mathcal{A}b^{\mathrm{op}} \times \mathcal{E} \longrightarrow \mathcal{E}.$$

In this case, for any abelian group M and any object E of $\mathcal{E}$, the object $[M,E]$ of $\mathcal{E}$ is characterized by

$$\mathrm{Hom}_{\mathcal{E}}(X,[M,E]) \simeq \mathrm{Hom}_{\mathcal{A}b}(M, \mathrm{Hom}_{\mathcal{E}}(X,E)) \qquad \forall X \in \mathrm{Ob}(\mathcal{E}).$$

Let us recall the following easy formulas:

Proposition 4.1.2. *(i) For any object E of $\mathcal{E}$, we have*

$$[\mathbb{Z},E] \simeq E.$$

(ii) For any object M of $\mathcal{A}b^{\mathcal{I}}$ and any object E of $\mathcal{E}$, we have

$$[\varinjlim_{i\in\mathcal{I}} M(i), E] \simeq \varprojlim_{i\in\mathcal{I}} [M(i),E].$$

(iii) For any abelian group M and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$[M, \varprojlim_{i\in\mathcal{I}} E(i)] \simeq \varprojlim_{i\in\mathcal{I}} [M,E(i)].$$

4.2 Derivation of the symbolic-Hom functor

In this section, $\mathcal{I}$ will denote a small category.

Lemma 4.2.1. *Let $\mathcal{E}$ be a quasi-abelian category with exact products. If P is a projective abelian group, then the functor*

$$[P,\cdot] : \mathcal{E} \to \mathcal{E}$$

is exact.

Proof. First, let us prove the result when P is a free abelian group, i.e.,

$$P = \mathbb{Z}^{(I)} = \bigoplus_{i \in I} \mathbb{Z}.$$

Consider a strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of $\mathcal{E}$. Since for any object E of $\mathcal{E}$,

$$[\mathbb{Z}^{(I)}, E] \simeq \prod_{i \in I} [\mathbb{Z}, E] \simeq \prod_{i \in I} E$$

and since products are exact, the sequence

$$0 \longrightarrow [\mathbb{Z}^{(I)}, E'] \longrightarrow [\mathbb{Z}^{(I)}, E] \longrightarrow [\mathbb{Z}^{(I)}, E''] \longrightarrow 0$$

is exact. Next, consider a projective abelian group P. We know that there is an exact sequence

$$0 \longrightarrow \ker q \longrightarrow \mathbb{Z}^{(I)} \xrightarrow{q} P \longrightarrow 0.$$

Since P is projective, this sequence splits and we have

$$[\mathbb{Z}^{(I)}, \cdot] \simeq [(\ker q) \oplus P, \cdot] \simeq [\ker q, \cdot] \oplus [P, \cdot].$$

The functor $[\mathbb{Z}^{(I)}, \cdot]$ being exact, the functor $[P, \cdot]$ is also exact. □

Lemma 4.2.2. *Let $\mathcal{E}$ be a complete additive category.*

(i) For any object S of $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$,we have

$$[\Pi(S), E]_{\mathcal{I}} \simeq [S, \mathrm{O}(E)]_{\mathrm{Ob}(\mathcal{I})} \simeq \prod_{i \in \mathcal{I}} [S(i), E(i)].$$

(ii) For any object M of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ and any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, we have

$$[M, \Pi(S)]_{\mathcal{I}} \simeq [\mathrm{O}(M), S]_{\mathrm{Ob}(\mathcal{I})} \simeq \prod_{i \in \mathcal{I}} [M(i), S(i)].$$

Proof. (i) Let X be an object of $\mathcal{E}$. First, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{E}}(X, [\Pi(S), E]_{\mathcal{I}}) &\simeq \mathrm{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(\Pi(S), h_X \circ E) \\ &\simeq \mathrm{Hom}_{\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}}(S, \mathrm{O}(h_X \circ E)) \qquad (*) \\ &\simeq \mathrm{Hom}_{\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}}(S, h_X \circ \mathrm{O}(E)) \\ &\simeq \mathrm{Hom}_{\mathcal{E}}(X, [S, \mathrm{O}(E)]_{\mathrm{Ob}(\mathcal{I})}), \end{aligned}$$

where the isomorphism (*) follows from Proposition 2.2.6. Next, we get

$$\begin{aligned}\operatorname{Hom}_{\mathcal{E}}(X,[S,\mathrm{O}(E)]_{\mathrm{Ob}(\mathcal{I})}) &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}}(S,h_X\circ \mathrm{O}(E))\\ &\simeq \prod_{i\in\mathcal{I}}\operatorname{Hom}_{\mathcal{A}b}(S(i),\operatorname{Hom}_{\mathcal{E}}(X,E(i)))\\ &\simeq \prod_{i\in\mathcal{I}}\operatorname{Hom}_{\mathcal{E}}(X,[S(i),E(i)])\\ &\simeq \operatorname{Hom}_{\mathcal{E}}(X,\prod_{i\in\mathcal{I}}[S(i),E(i)]).\end{aligned}$$

(ii) Let X be an object of $\mathcal{E}$. On one hand, we have successively

$$\begin{aligned}\operatorname{Hom}_{\mathcal{E}}(X,[M,\Pi(S)]_{\mathcal{I}}) &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(M,h_X\circ \Pi(S))\\ &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(M,\Pi(h_X\circ S))\\ &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}}(\mathrm{O}(M),h_X\circ S)\\ &\simeq \operatorname{Hom}_{\mathcal{E}}(X,[\mathrm{O}(M),S]_{\mathrm{Ob}(\mathcal{I})}).\end{aligned}$$

On the other hand, we get

$$\begin{aligned}\operatorname{Hom}_{\mathcal{E}}(X,[\mathrm{O}(M),S]_{\mathrm{Ob}(\mathcal{I})}) &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}}(\mathrm{O}(M),h_X\circ S)\\ &\simeq \prod_{i\in\mathcal{I}}\operatorname{Hom}_{\mathcal{A}b}(M(i),\operatorname{Hom}_{\mathcal{E}}(X,S(i)))\\ &\simeq \prod_{i\in\mathcal{I}}\operatorname{Hom}_{\mathcal{E}}(X,[M(i),S(i)])\\ &\simeq \operatorname{Hom}_{\mathcal{E}}(X,\prod_{i\in\mathcal{I}}[M(i),S(i)]).\end{aligned}$$

□

Lemma 4.2.3. *Let $\mathcal{E}$ be a quasi-abelian category with exact products. If P is a projective object of $\mathcal{A}b^{\mathcal{I}}$, then the functor*

$$[P,\cdot]_{\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{E}$$

is exact.

Proof. First, let us prove the result when $P = \amalg(S)$ where S is a projective object of $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$. Consider a strictly exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. Since for any $i \in \mathcal{I}$, $S(i)$ is projective, by Lemma 4.2.1, the functor

$$[S(i), \cdot] : \mathcal{E} \longrightarrow \mathcal{E}$$

is exact. Moreover, since for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$,

$$[\mathrm{II}(S), E]_{\mathcal{I}} \simeq \prod_{i \in \mathcal{I}} [S(i), E(i)]$$

and since products are exact, the sequence

$$0 \longrightarrow [\mathrm{II}(S), E']_{\mathcal{I}} \longrightarrow [\mathrm{II}(S), E]_{\mathcal{I}} \longrightarrow [\mathrm{II}(S), E'']_{\mathcal{I}} \longrightarrow 0$$

is exact. Next, consider a projective object P of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$. We know that there is an epimorphism

$$q : \mathrm{II}(S) \longrightarrow P$$

where S is a projective object of $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$. Since P is projective, the exact sequence

$$0 \longrightarrow \ker q \longrightarrow \mathrm{II}(S) \xrightarrow{q} P \longrightarrow 0$$

splits and we have

$$[\mathrm{II}(S), \cdot]_{\mathcal{I}} \simeq [(\ker q) \oplus P, \cdot]_{\mathcal{I}} \simeq [\ker q, \cdot]_{\mathcal{I}} \oplus [P, \cdot]_{\mathcal{I}}.$$

The functor $[\mathrm{II}(S), \cdot]_{\mathcal{I}}$ being exact, the functor $[P, \cdot]_{\mathcal{I}}$ is also exact. $\square$

Proposition 4.2.4. *Let $\mathcal{E}$ be a quasi-abelian category with exact products. The functor*

$$[\cdot, \cdot]_{\mathcal{I}} : (\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})^{\mathrm{op}} \times \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

has a right derived functor

$$\mathrm{R}\,[\cdot, \cdot]_{\mathcal{I}} : (D^-(\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}))^{\mathrm{op}} \times D^+(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}) \longrightarrow D^+(\mathcal{E}).$$

Proof. Let us show that if $\mathcal{P}$ is the full subcategory of projective objects of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$, then the pair $(\mathcal{P}, \mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$ is $[\cdot, \cdot]_{\mathcal{I}}$-acyclic.

First, consider an object P of $\mathcal{P}$. Since the functor

$$[P, \cdot]_{\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

is exact, the category $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ is $[P, \cdot]_{\mathcal{I}}$-injective.

Next, consider an object E of $\mathrm{Ob}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$ and let us show that $\mathcal{P}$ is $[\cdot, E]_{\mathcal{I}}$-projective.

(i) For any object M of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$, there is a projective object P of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ and an epimorphism $P \longrightarrow M$.

(ii) If

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is an exact sequence of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$, where P and P'' are two objects of $\mathcal{P}$, then we know that P' is an object of $\mathcal{P}$.

(iii) If

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$$

is an exact sequence of $\mathcal{P}$, then it splits. It follows that the sequence

$$0 \longrightarrow [P'', E]_{\mathcal{I}} \longrightarrow [P, E]_{\mathcal{I}} \longrightarrow [P', E]_{\mathcal{I}} \longrightarrow 0$$

of $\mathcal{E}$ splits and that it is exact.

Since the functor $[\cdot,\cdot]_{\mathcal{I}}$ is left exact, the conclusion follows from Proposition 1.4.7.

□

4.3 Links with the derived projective limit functor

Lemma 4.3.1. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a complete additive category. If*

$$\mathbb{Z}_{\mathcal{I}} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{A}b$$

is the constant functor which associates to any $i \in \mathcal{I}$ the abelian group $\mathbb{Z}$, then for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \simeq \varprojlim_{i \in \mathcal{I}} E(i).$$

Proof. For any object X of $\mathcal{E}$, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{E}}(X, [\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}}) &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(\mathbb{Z}_{\mathcal{I}}, h_X \circ E) \\
&\simeq \varprojlim_{i \in \mathcal{I}} (h_X \circ E)(i) \\
&\simeq \varprojlim_{i \in \mathcal{I}} \operatorname{Hom}_{\mathcal{E}}(X, E(i)) \\
&\simeq \operatorname{Hom}_{\mathcal{E}}(X, \varprojlim_{i \in \mathcal{I}} E(i)).
\end{aligned}$$

□

Proposition 4.3.2. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with exact products. For any object M of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ and any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, we have*

$$\mathrm{R}\,[M, \Pi(S)]_{\mathcal{I}} \simeq \mathrm{R}\,[\mathrm{O}(M), S]_{\mathrm{Ob}(\mathcal{I})}\,.$$

Proof. We know that there is a projective resolution $P.$ of M in $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ such that for any $n \geq 0$

$$P_n = \amalg(S_n)$$

where S_n is a projective object of $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$. Then, we have

$$\mathrm{R}\,[M, \Pi(S)]_{\mathcal{I}} \simeq [P., \Pi(S)]_{\mathcal{I}} \simeq [\mathrm{O}(P.), S]_{\mathrm{Ob}(\mathcal{I})}$$

where the second isomorphism follows from Lemma 4.2.2. Since for any $n \geq 0$ and any $i \in \mathcal{I}$, we have

$$\mathrm{O}(P_n)(i) = P_n(i) = \amalg(S_n)(i) = \coprod_{i \xrightarrow{\alpha} j} S_n(j),$$

$\mathrm{O}(P_n)$ is projective in $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$. The functor

$$\mathrm{O} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$$

being exact, $\mathrm{O}(P.)$ is a projective resolution of $\mathrm{O}(M)$. It follows that

$$\begin{aligned}\mathrm{R}\,[\mathrm{O}(M), S]_{\mathrm{Ob}(\mathcal{I})} &\simeq [\mathrm{O}(P.), S]_{\mathrm{Ob}(\mathcal{I})} \\ &\simeq \mathrm{R}\,[M, \Pi(S)]_{\mathcal{I}}\,.\end{aligned}$$

□

Corollary 4.3.3. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with exact products. For any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, we have*

$$\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, \Pi(S)]_{\mathcal{I}} \simeq [\mathbb{Z}_{\mathcal{I}}, \Pi(S)]_{\mathcal{I}}.$$

Proof. The object $\mathrm{O}(\mathbb{Z}_{\mathcal{I}})$ of $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$ is projective since for any $i \in \mathcal{I}$,

$$\mathrm{O}(\mathbb{Z}_{\mathcal{I}})(i) = \mathbb{Z}_{\mathcal{I}}(i) = \mathbb{Z}.$$

It follows that

$$\begin{aligned}\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, \Pi(S)]_{\mathcal{I}} &\simeq \mathrm{R}\,[\mathrm{O}(\mathbb{Z}_{\mathcal{I}}), S]_{\mathrm{Ob}(\mathcal{I})} \\ &\simeq [\mathrm{O}(\mathbb{Z}_{\mathcal{I}}), S]_{\mathrm{Ob}(\mathcal{I})} \\ &\simeq [\mathbb{Z}_{\mathcal{I}}, \Pi(S)]_{\mathcal{I}}\end{aligned}$$

where the last isomorphism follows from Lemma 4.2.2. □

Proposition 4.3.4. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with exact products. Then, the functor*

$$\varprojlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

is right derivable and for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}}\,.$$

Proof. First, let us remark that, since the functor

$$[\cdot,\cdot]_{\mathcal{I}} : (\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})^{\mathrm{op}} \times \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

is left exact, we have

$$\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \simeq [\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}}$$

for an object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, if and only if

$$LH^k\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \simeq 0 \qquad \forall k > 0.$$

Next, let us show that the family

$$\mathcal{F} = \{E \in \mathrm{Ob}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}) : \mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \simeq [\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}}\}$$

is $\varprojlim_{i\in\mathcal{I}}$-injective.

(i) Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. By Proposition 2.3.2, there is a strict monomorphism

$$E \longrightarrow \Pi(\mathrm{O}(E))$$

and by Corollary 4.3.3, $\Pi(\mathrm{O}(E))$ belongs to $\mathcal{F}$.

(ii) Consider a strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ where E' and E belong to $\mathcal{F}$. This sequence gives rise to the long exact sequence

$$\begin{aligned}
&0 \longrightarrow [\mathbb{Z}_{\mathcal{I}}, E']_{\mathcal{I}} \longrightarrow [\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \longrightarrow [\mathbb{Z}_{\mathcal{I}}, E'']_{\mathcal{I}} \longrightarrow \\
&\longrightarrow LH^1\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E']_{\mathcal{I}} \longrightarrow LH^1\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \longrightarrow LH^1\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E'']_{\mathcal{I}} \longrightarrow \\
&\longrightarrow LH^2\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E']_{\mathcal{I}} \longrightarrow LH^2\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \longrightarrow LH^2\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E'']_{\mathcal{I}} \cdots
\end{aligned}$$

of $C^+(\mathcal{E})$. Since E' and E are objects of $\mathcal{F}$,

$$LH^k\mathrm{R}\,[\mathbb{Z}_\mathcal{I}, E']_\mathcal{I} = LH^k\mathrm{R}\,[\mathbb{Z}_\mathcal{I}, E]_\mathcal{I} = 0 \qquad \forall k > 0$$

and it follows that

$$LH^k\mathrm{R}\,[\mathbb{Z}_\mathcal{I}, E'']_\mathcal{I} = 0 \qquad \forall k > 0.$$

Hence, E'' belongs to $\mathcal{F}$.

Moreover, by Lemma 4.3.1, for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$[\mathbb{Z}_\mathcal{I}, E]_\mathcal{I} \simeq \varprojlim_{i\in\mathcal{I}} E(i),$$

so the preceding long exact sequence shows that the sequence

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} E'(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} E''(i) \longrightarrow 0$$

is exact. Consequently, $\mathcal{F}$ is $\varprojlim_{i\in\mathcal{I}}$-injective and the functor

$$\varprojlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

is right derivable.

It follows from Proposition 2.3.2 that any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ has a resolution $I^\cdot$ by projective systems of product type. Assume that for any n, I^n is of the form

$$I^n \simeq \Pi(S^n).$$

Then, for any $n \geq 0$, we have

$$\begin{aligned}\mathrm{R}\,[\mathbb{Z}_\mathcal{I}, I^n]_\mathcal{I} &\simeq \mathrm{R}\,[\mathbb{Z}_\mathcal{I}, \Pi(S^n)]_\mathcal{I}\\ &\simeq [\mathbb{Z}_\mathcal{I}, \Pi(S^n)]_\mathcal{I}\\ &\simeq [\mathbb{Z}_\mathcal{I}, I^n]_\mathcal{I}.\end{aligned}$$

It follows that

$$\mathrm{R}\,[\mathbb{Z}_\mathcal{I}, I^\cdot]_\mathcal{I} \simeq [\mathbb{Z}_\mathcal{I}, I^\cdot]_\mathcal{I}.$$

Therefore, we get

$$\begin{aligned}\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) &\simeq \varprojlim_{i\in\mathcal{I}} I^\cdot(i)\\ &\simeq [\mathbb{Z}_\mathcal{I}, I^\cdot]_\mathcal{I}\\ &\simeq \mathrm{R}\,[\mathbb{Z}_\mathcal{I}, I^\cdot]_\mathcal{I}\\ &\simeq \mathrm{R}\,[\mathbb{Z}_\mathcal{I}, E]_\mathcal{I}\end{aligned}$$

where the first isomorphism follows from the first part of the proof and the second from Lemma 4.3.1. □

Proposition 4.3.5. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with exact products. Then, for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have a canonical isomorphism*

$$R^{\cdot}(\mathcal{I}, E) \simeq [R_{\cdot}(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}}$$

in $C^+(\mathcal{E})$ making the diagram

$$\begin{tikzcd} \mathrm{R}\varprojlim\limits_{i\in\mathcal{I}} E(i) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \arrow[d, "\wr"] \\ R^{\cdot}(\mathcal{I}, E) \arrow[r, "\sim"] & [R_{\cdot}(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}} \end{tikzcd}$$

commutative.

Proof. For any object E of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ and any $n \geq 0$, we have the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(R_n(\mathbb{Z}_{\mathcal{I}}), E) &\simeq \mathrm{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(\mathrm{II}(S_n(\mathbb{Z}_{\mathcal{I}})), E) \\ &\simeq \mathrm{Hom}_{\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}}(S_n(\mathbb{Z}_{\mathcal{I}}), \mathrm{O}(E)) \\ &\simeq \prod_{i_0\in\mathcal{I}} \mathrm{Hom}_{\mathcal{A}b}(S_n(\mathbb{Z}_{\mathcal{I}})(i_0), \mathrm{O}(E)(i_0)) \\ &\simeq \prod_{i_0\in\mathcal{I}} \mathrm{Hom}_{\mathcal{A}b}\Big(\coprod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} \mathbb{Z}_{\mathcal{I}}(i_n), E(i_0)\Big) \\ &\simeq \prod_{i_0\in\mathcal{I}} \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} \mathrm{Hom}_{\mathcal{A}b}(\mathbb{Z}, E(i_0)) \\ &\simeq \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} E(i_0) \\ &\simeq R^n(\mathcal{I}, E). \end{aligned}$$

A direct computation shows that these isomorphisms are compatible with the differentials. Hence, we have

$$\mathrm{Hom}\,(R_{\cdot}(\mathbb{Z}_{\mathcal{I}}), E) \simeq R^{\cdot}(\mathcal{I}, E).$$

Then, for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and any object X of $\mathcal{E}$, we have successively

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{E}}(X, [R_n(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}}) &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(R_n(\mathbb{Z}_{\mathcal{I}}), h_X \circ E)\\
&\simeq R^n(\mathcal{I}, h_X \circ E)\\
&\simeq \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} \operatorname{Hom}_{\mathcal{E}}(X, E(i_0))\\
&\simeq \operatorname{Hom}_{\mathcal{E}}(X, \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} E(i_0))\\
&\simeq \operatorname{Hom}_{\mathcal{E}}(X, R^n(\mathcal{I}, E)).
\end{aligned}$$

It follows that

$$[R_n(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}} \simeq R^n(\mathcal{I}, E)$$

and that

$$[R_\cdot(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}} \simeq R^\cdot(\mathcal{I}, E).$$

Since

$$R_n(\mathbb{Z}_{\mathcal{I}}) \simeq \amalg(S_n(\mathbb{Z}_{\mathcal{I}}))$$

and

$$S_n(\mathbb{Z}_{\mathcal{I}})(i_0) \simeq \coprod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} \mathbb{Z}(i_n),$$

it follows from the dual of Proposition 2.3.1 that $R_n(\mathbb{Z}_{\mathcal{I}})$ is a projective object of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$. Together with the fact that $R_\cdot(\mathbb{Z}_{\mathcal{I}})$ is a resolution of $\mathbb{Z}_{\mathcal{I}}$, this explains the second vertical isomorphism of the diagram in our statement.

The commutativity of the diagram follows directly from the construction of the various morphisms. □

4.4 Index restriction

Proposition 4.4.1. *Let $\mathcal{E}$ be a complete additive category and let $J : \mathcal{J} \to \mathcal{I}$ be a functor between small categories. For any object M of $\mathcal{A}b^{\mathcal{J}^{\mathrm{op}}}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have*

$$[J_+(M), E]_{\mathcal{I}} \simeq [M, J^+(E)]_{\mathcal{J}}.$$

Proof. This isomorphism follows from Proposition 2.4.9. As a matter of fact, for any object X of $\mathcal{E}$, we have successively

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{E}}(X, [J_+(M), E]_{\mathcal{I}}) &\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(J_+(M), h_X \circ E)\\
&\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{J}^{\mathrm{op}}}}(M, J^+(h_X \circ E))\\
&\simeq \operatorname{Hom}_{\mathcal{A}b^{\mathcal{J}^{\mathrm{op}}}}(M, h_X \circ J^+(E))\\
&\simeq \operatorname{Hom}_{\mathcal{E}}(X, [M, J^+(E)]_{\mathcal{J}}).
\end{aligned}$$

□

Lemma 4.4.2. *Let $\mathcal{E}$ be a cocomplete category and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small categories. If P is a projective object of $\mathcal{E}^{\mathcal{J}^{\mathrm{op}}}$, then $J_+(P)$ is a projective object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.*

Proof. Consider a strictly exact sequence

$$E' \longrightarrow E \longrightarrow E''$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. Since J^+ is exact and since P is projective , the sequence

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{J}^{\mathrm{op}}}}(P, J^+(E')) \longrightarrow \mathrm{Hom}_{\mathcal{E}^{\mathcal{J}^{\mathrm{op}}}}(P, J^+(E)) \longrightarrow \mathrm{Hom}_{\mathcal{E}^{\mathcal{J}^{\mathrm{op}}}}(P, J^+(E''))$$

is exact. The conclusion follows from Proposition 2.4.9. □

Proposition 4.4.3. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small categories. For any object M of $\mathcal{A}b^{\mathcal{J}^{\mathrm{op}}}$ and any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have*

$$\mathrm{R}\,[LJ_+(M), E]_{\mathcal{I}} \simeq \mathrm{R}\left[M, J^+(E)\right]_{\mathcal{J}}.$$

Proof. If $P_\cdot$ is a projective resolution of M, then we have successively

$$\begin{aligned}
\mathrm{R}\left[M, J^+(E)\right]_{\mathcal{J}} &\simeq [P_\cdot, J^+(E)]_{\mathcal{J}} \\
&\simeq [J_+(P_\cdot), E]_{\mathcal{I}} \\
&\simeq \mathrm{R}\,[J_+(P_\cdot), E]_{\mathcal{I}} \\
&\simeq \mathrm{R}\,[LJ_+(M), E]_{\mathcal{I}}.
\end{aligned}$$

□

Proposition 4.4.4. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small categories. The canonical isomorphism*

$$\mathrm{id} : \mathbb{Z}_{\mathcal{J}} \longrightarrow J^+(\mathbb{Z}_{\mathcal{I}}) = \mathbb{Z}_{\mathcal{J}}$$

induces by adjunction a canonical morphism

$$w : LJ_+(\mathbb{Z}_{\mathcal{J}}) \longrightarrow \mathbb{Z}_{\mathcal{I}}$$

which makes the diagram

$$\begin{tikzcd}
LJ_+(\mathbb{Z}_{\mathcal{J}}) \arrow[r, "w"] & \mathbb{Z}_{\mathcal{I}} \\
J_+(R_\cdot(J^+(\mathbb{Z}_{\mathcal{I}}))) \arrow[u, "\wr"] \arrow[r, "\tilde{\rho}_\cdot(J, \mathbb{Z}_{\mathcal{I}})"'] & R_\cdot(\mathbb{Z}_{\mathcal{I}}) \arrow[u, "\wr", "\epsilon_\cdot(\mathbb{Z}_{\mathcal{I}})"']
\end{tikzcd}$$

commutative in $D^-(\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})$.

Proof. Recall that w is the composition of

$$u : LJ_+(\mathbb{Z}_\mathcal{J}) \longrightarrow J_+(\mathbb{Z}_\mathcal{J}),$$

and the morphism

$$v : J_+(\mathbb{Z}_\mathcal{J}) \longrightarrow \mathbb{Z}_\mathcal{I}$$

defined as the image of $\mathrm{id} : \mathbb{Z}_\mathcal{J} \longrightarrow J^+(\mathbb{Z}_\mathcal{I}) = \mathbb{Z}_\mathcal{J}$ by the adjunction

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{J}^{\mathrm{op}}}}(\mathbb{Z}_\mathcal{J}, J^+(\mathbb{Z}_\mathcal{I})) \longrightarrow \mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(J_+(\mathbb{Z}_\mathcal{J}), \mathbb{Z}_\mathcal{I}).$$

Since $R.(\mathbb{Z}_\mathcal{J})$ is a projective resolution of $\mathbb{Z}_\mathcal{J}$, we have

$$J_+(R.(J^+(\mathbb{Z}_\mathcal{I}))) \simeq J_+(R.(\mathbb{Z}_\mathcal{J})) \simeq LJ_+(\mathbb{Z}_\mathcal{J})$$

in $D^-(\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})$. Consider the diagram

$$\begin{tikzcd}
J_+(R.(\mathbb{Z}_\mathcal{J})) \arrow[r, "\tilde{\rho}.(J,\mathbb{Z}_\mathcal{I})"] \arrow[d, "\wr"] \arrow[dd, bend right=60, "J_+(\epsilon.(\mathbb{Z}_\mathcal{J}))"'] & R.(\mathbb{Z}_\mathcal{I}) \arrow[d, "\wr\, \epsilon.(\mathbb{Z}_\mathcal{I})"] \\
LJ_+(\mathbb{Z}_\mathcal{J}) \arrow[r, "w"] \arrow[d, "u"'] & \mathbb{Z}_\mathcal{I} \arrow[d, "\wr"] \\
J_+(\mathbb{Z}_\mathcal{J}) \arrow[r, "v"'] & \mathbb{Z}_\mathcal{I}
\end{tikzcd}$$

It follows from the construction of w that the lower square is commutative. Since a direct computation shows that the diagram

$$\begin{tikzcd}
J_+(R.(\mathbb{Z}_\mathcal{J})) \arrow[r, "\tilde{\rho}.(J,\mathbb{Z}_\mathcal{I})"] \arrow[d, "J_+(\epsilon.(\mathbb{Z}_\mathcal{J}))"'] & R.(\mathbb{Z}_\mathcal{I}) \arrow[d, "\wr\, \epsilon.(\mathbb{Z}_\mathcal{I})"] \\
J_+(\mathbb{Z}_\mathcal{J}) \arrow[r, "v"'] & \mathbb{Z}_\mathcal{I}
\end{tikzcd}$$

is commutative in $C^-(\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})$ and the conclusion follows. $\square$

Proposition 4.4.5. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small categories. Then, for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, the canonical diagram*

$$\begin{tikzcd}
\mathrm{R}\,[\mathbb{Z}_\mathcal{I}, E]_\mathcal{I} \arrow[r, "\sim"] \arrow[d, "\mathrm{R}[w,E]"'] & \mathrm{R}\varprojlim\limits_{i\in\mathcal{I}} E(i) \arrow[dd] \\
\mathrm{R}\,[LJ_+(\mathbb{Z}_\mathcal{J}), E]_\mathcal{I} \arrow[d, "\wr"] & \\
\mathrm{R}\left[\mathbb{Z}_\mathcal{J}, J^+(E)\right]_\mathcal{J} \arrow[r, "\sim"] & \mathrm{R}\varprojlim\limits_{j\in\mathcal{J}} J^+(E)(j)
\end{tikzcd}$$

is commutative.

Proof. Consider the following diagram:

$$
\begin{tikzcd}
\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \arrow[rrr, "\sim"] \arrow[dd] \arrow[dr, "\sim"] & & & \mathrm{R}\varprojlim\limits_{i\in\mathcal{I}} E(i) \arrow[dl, "\sim"'] \arrow[dddd] \\
& {[R.(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}}} \arrow[r, "\sim"] \arrow[d] \arrow[urr, phantom, "(2)"] & R^{\cdot}(\mathcal{I}, E) \arrow[dd] \arrow[dr, phantom, "(6)"] & \\
\mathrm{R}\,[LJ_{+}(\mathbb{Z}_{\mathcal{J}}), E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[dd, "\wr"] \arrow[ur, phantom, "(3)"] \arrow[dr, phantom, "(4)"] & {[J_{+}(R.(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}}} \arrow[d, "\wr"] \arrow[r, phantom, "(1)"] & & \\
& {[R.(\mathbb{Z}_{\mathcal{J}}), J^{+}(E)]_{\mathcal{J}}} \arrow[r, "\sim"] \arrow[dl, "\sim"] \arrow[drr, phantom, "(5)"] & R^{\cdot}(\mathcal{J}, J^{+}(E)) \arrow[dr, "\sim"] & \\
\mathrm{R}\left[\mathbb{Z}_{\mathcal{J}}, J^{+}(E)\right]_{\mathcal{J}} \arrow[rrr, "\sim"] & & & \mathrm{R}\varprojlim\limits_{j\in\mathcal{J}} J^{+}(E)(j)
\end{tikzcd}
$$

Clearly, the result will be established if we prove that the subdiagrams (1)–(6) commute. We know already from Proposition 3.4.2 that diagram (6) commutes. Moreover, Proposition 4.3.5 shows that diagrams (2) and (5) are also commutative. Since the commutativity of (1), (3) and (4) follows from the lemmas below, the proof is complete. $\square$

Lemma 4.4.6. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \to \mathcal{I}$ be a functor between two small categories. For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, the canonical diagram*

$$
\begin{tikzcd}
\mathrm{R}\,[LJ_{+}(\mathbb{Z}_{\mathcal{J}}), E]_{\mathcal{I}} \arrow[r, "\sim"] & {[J_{+}(R.(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}}} \\
\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[u, "{\mathrm{R}[w,E]_{\mathcal{I}}}"] & {[R.(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}}} \arrow[u, "{[\tilde{\rho}.(J,\mathbb{Z}_{\mathcal{I}}),E]_{\mathcal{I}}}"']
\end{tikzcd}
$$

is commutative in $D^{+}(\mathcal{E})$.

Proof. Since $R.(\mathbb{Z}_{\mathcal{I}})$ is a projective resolution of $\mathbb{Z}_{\mathcal{I}}$, we have

$$\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \simeq [R.(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}}.$$

Moreover, using Lemma 4.4.2, we see that

$$\mathrm{R}\,[LJ_+(\mathbb{Z}_{\mathcal{J}}), E]_{\mathcal{I}} \simeq \mathrm{R}\,[J_+(R_\cdot(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}} \simeq [J_+(R_\cdot(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}}.$$

This explains the horizontal isomorphism. Now, consider the diagram

$$\begin{tikzcd}
\mathrm{R}\,[LJ_+(\mathbb{Z}_{\mathcal{J}}), E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[dr, phantom, "(1)"] & \mathrm{R}\,[J_+(R_\cdot(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[dr, phantom, "(2)"] & [J_+(R_\cdot(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}} \\
\mathrm{R}\,[\mathbb{Z}_{\mathcal{I}}, E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[u, "{\mathrm{R}[w,E]_{\mathcal{I}}}"] & \mathrm{R}\,[R_\cdot(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[u, "{\mathrm{R}[\tilde{\rho}_\cdot(J,\mathbb{Z}_{\mathcal{I}}),E]_{\mathcal{I}}}"] & [R_\cdot(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}} \arrow[u, "{[\tilde{\rho}_\cdot(J,\mathbb{Z}_{\mathcal{I}}),E]_{\mathcal{I}}}"]
\end{tikzcd}$$

By Proposition 4.4.4, the square (1) is commutative. The square (2) being clearly commutative, the conclusion follows. □

Lemma 4.4.7. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \to \mathcal{I}$ be a functor between two small categories. For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, the canonical diagram*

$$\begin{tikzcd}
\mathrm{R}\left[\mathbb{Z}_{\mathcal{J}}, J^+(E)\right]_{\mathcal{J}} \arrow[r, "\sim"] & {[R_\cdot(\mathbb{Z}_{\mathcal{J}}), J^+(E)]_{\mathcal{J}}} \\
\mathrm{R}\,[LJ_+(\mathbb{Z}_{\mathcal{J}}), E]_{\mathcal{I}} \arrow[r, "\sim"] \arrow[u, "\wr"] & {[J_+(R_\cdot(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}}} \arrow[u, "\wr"]
\end{tikzcd}$$

is commutative in $D^+(\mathcal{E})$.

Proof. This follows directly from the construction in the proof of Proposition 4.4.3 if one keeps in mind that $R_\cdot(\mathbb{Z}_{\mathcal{J}})$ is a projective resolution of $\mathbb{Z}_{\mathcal{J}}$. □

Lemma 4.4.8. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \to \mathcal{I}$ be a functor between two small categories. For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, the diagram*

$$\begin{tikzcd}
{[R_\cdot(\mathbb{Z}_{\mathcal{I}}), E]_{\mathcal{I}}} \arrow[r, "\sim"] \arrow[d, "{[\tilde{\rho}_\cdot(J,\mathbb{Z}_{\mathcal{I}}),E]}"'] & R^\cdot(\mathcal{I}, E) \arrow[dd, "{R^\cdot(J,E)}"] \\
{[J_+(R_\cdot(\mathbb{Z}_{\mathcal{J}})), E]_{\mathcal{I}}} \arrow[d, "\wr"] & \\
{[R_\cdot(\mathbb{Z}_{\mathcal{J}}), J^+(E)]_{\mathcal{J}}} \arrow[r, "\sim"] & R^\cdot(\mathcal{J}, J^+(E))
\end{tikzcd}$$

is commutative in $C^+(\mathcal{E})$.

Proof. Working as in the proof of Proposition 4.3.5, we reduce the problem to show that for any object E of $\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}$ and any $n \geq 0$ the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(R_n(\mathbb{Z}_{\mathcal{I}}), E) \arrow[rr, "\sim"] \arrow[d, "\operatorname{Hom}(\tilde{\rho}_n(J,\mathbb{Z}_{\mathcal{I}}),E)"'] & & R^n(\mathcal{I}, E) \arrow[dd, "R^n(J,E)"] \\
\operatorname{Hom}_{\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}}}(J_+(R_n(\mathbb{Z}_{\mathcal{J}})), E) \arrow[d, "\wr"] & & \\
\operatorname{Hom}_{\mathcal{A}b^{\mathcal{J}^{\mathrm{op}}}}(R_n(\mathbb{Z}_{\mathcal{J}}), J^+(E)) \arrow[rr, "\sim"] & & R^n(\mathcal{J}, J^+(E))
\end{tikzcd}$$

is commutative. This is done by direct computation. □

Lemma 4.4.9. *Let $J : \mathcal{J} \to \mathcal{I}$ be a functor between two small categories and let M be an object of $\mathcal{A}b^{\mathcal{J}^{\mathrm{op}}}$. Then, for any object i of $\mathcal{I}$, we have the isomorphism*

$$LJ_+(M)(i) \simeq \mathrm{L} \varinjlim_{(f,j)\in i\downarrow J} M(j).$$

Proof. We know that M has a projective resolution $I.$ such that for any $n \geq 0$,

$$I_n = \amalg(S_n)$$

where S_n is a projective object of $\mathcal{A}b^{\mathrm{Ob}(\mathcal{I})}$. Then,

$$LJ_+(M) \simeq J_+(I.).$$

It follows that for any $i \in \mathcal{I}$,

$$\begin{aligned} LJ_+(M)(i) &\simeq J_+(I.)(i) \\ &\simeq \varinjlim_{(f,j)\in i\downarrow J} (I. \circ J^{i,\mathrm{op}})(f,j). \end{aligned}$$

To conclude, it is sufficient to show that

$$\mathrm{L} \varinjlim_{(f,j)\in i\downarrow J} (M \circ J^{i,\mathrm{op}})(f,j) \simeq \varinjlim_{(f,j)\in i\downarrow J} (I. \circ J^{i,\mathrm{op}})(f,j).$$

This will be the case if $I. \circ J^{i,\mathrm{op}}$ is a $\varinjlim_{(f,j)\in i\downarrow J}$-acyclic resolution of $M \circ J^{i,\mathrm{op}}$. Since for any $n \geq 0$, and any object (f,j) of $i \downarrow J$,

$$(I_n \circ J^{i,\mathrm{op}})(f,j) = I_n(j),$$

$I. \circ J^{i,\mathrm{op}}$ is a resolution of $M \circ J^{i,\mathrm{op}}$. Now, let us show that for any $n \geq 0$, $I_n \circ J^{i,\mathrm{op}}$ is of coproduct type. Since

$$I_n \circ J^{i,\mathrm{op}} = \amalg(S_n) \circ J^{i,\mathrm{op}},$$

it is sufficient to show that

$$\amalg(S_n) \circ J^{i,\mathrm{op}} \simeq \amalg(S_n \circ \mathrm{O}(J^{i,\mathrm{op}})).$$

Let (f,j) be an object of $i \downarrow J$. On one hand, we have

$$(\amalg(S_n) \circ J^{i,\mathrm{op}})(f,j) = \amalg(S_n)(j) = \coprod_{j \xrightarrow{\beta} j'} S_n(j').$$

On the other hand, we have

$$\amalg(S_n \circ \mathrm{O}(J^{i,\mathrm{op}}))(f,j) = \coprod_{(f,j) \xrightarrow{\beta} (f',j')} (S_n \circ \mathrm{O}(J^{i,\mathrm{op}}))(f',j') = \coprod_{(f,j) \xrightarrow{\beta} (f',j')} S_n(j').$$

Consider

$$u : \coprod_{j \xrightarrow{\beta} j'} S_n(j') \longrightarrow \coprod_{(f,j) \xrightarrow{\beta} (f',j')} S_n(j')$$

defined by

$$u \circ s_{j \xrightarrow{\beta} j'} = s_{(f,j) \xrightarrow{\beta} (J(\beta) \circ f, j')}$$

for any morphism $\beta : j \longrightarrow j'$ of $\mathcal{J}$ and consider

$$v : \coprod_{(f,j) \xrightarrow{\beta} (f',j')} S_n(j') \longrightarrow \coprod_{j \xrightarrow{\beta} j'} S_n(j')$$

defined by

$$v \circ s_{(f,j) \xrightarrow{\beta} (f',j')} = s_{j \xrightarrow{\beta} j'}$$

for any morphism $\beta : (f,j) \longrightarrow (f',j')$ of $i \downarrow J$. Of course, u and v are inverse one of each other. It follows that

$$\amalg(S_n) \circ J^{i,\mathrm{op}} \simeq \amalg(S_n \circ \mathrm{O}(J^{i,\mathrm{op}})).$$

and that $I_n \circ J^{i,\mathrm{op}}$ is $\varinjlim\limits_{(f,j) \in i \downarrow J}$-acyclic. $\square$

Proposition 4.4.10. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between small categories. Consider an object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. Then, the canonical morphism*

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} J^{+}(E)(j)$$

is an isomorphism in $D^{+}(\mathcal{E})$, if the canonical morphism

$$w : LJ_{+}(\mathbb{Z}_{\mathcal{J}}) \longrightarrow \mathbb{Z}_{\mathcal{I}}$$

is an isomorphism in $D^{-}(\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})$. This condition will be satisfied if and only if

$$\mathrm{L} \varinjlim_{(f,j)\in i\downarrow J} \mathbb{Z}_{\mathcal{J}}(j) \simeq \mathbb{Z}_{\mathcal{I}}(i)$$

for any $i \in \mathcal{I}$.

Proof. The first part of the result follows directly from Proposition 4.4.5.

As for the second part, it follows from the fact that w is an isomorphism of $D^{-}(\mathcal{A}b^{\mathcal{I}^{\mathrm{op}}})$ if and only if $w(i)$ is an isomorphism of $D^{-}(\mathcal{A}b)$ for any $i \in \mathcal{I}$ combined with Lemma 4.4.9. □

5 Derived projective limits and cofinality

5.1 Cofinal index restriction

Definition 5.1.1. Let $\Phi : \mathcal{I} \longrightarrow \mathcal{J}$ be a functor between small filtering categories. We will call Φ *cofinal* if it has the following properties:

(a) for any $j \in \mathcal{J}$, there is $i \in \mathcal{I}$ and a morphism

$$\alpha : j \longrightarrow \Phi(i),$$

(b) if $j \in \mathcal{J}$, if $i \in \mathcal{I}$ and if

$$j \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Phi(i)$$

is a pair of morphisms in $\mathcal{J}$, there is a morphism

$$\gamma : i \longrightarrow i'$$

of $\mathcal{I}$ such that

$$\Phi(\gamma) \circ \alpha = \Phi(\gamma) \circ \beta.$$

Remark 5.1.2. Let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a functor between two small filtering categories. Then, J is cofinal if and only if the category $i \downarrow J$ is non empty and filtering for any $i \in \mathcal{I}$.

Proposition 5.1.3. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a cofinal functor between two small filtering categories. For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have the isomorphism of $D^+(\mathcal{E})$*

$$\mathrm{R}\varprojlim_{j\in\mathcal{J}} J^+(E)(j) \simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i).$$

In particular,

$$R^{\cdot}(J,E) : R^{\cdot}(\mathcal{I},E) \longrightarrow R^{\cdot}(\mathcal{J},J^+(E))$$

is an isomorphism of $D^+(\mathcal{E})$.

Proof. Since the functor J is cofinal, the category $i \downarrow J$ is non empty and filtering. For any $i \in \mathcal{I}$, consider the functor

$$\mathbb{Z}_{\mathcal{J}} \circ J^{i,\mathrm{op}} : (i \downarrow J)^{\mathrm{op}} \longrightarrow \mathcal{A}b.$$

Since filtering inductive limits are exact in $\mathcal{A}b$, we have

$$\mathrm{L}\varinjlim_{(f,j)\in i\downarrow J} \mathbb{Z}_{\mathcal{J}} \circ J^{i,\mathrm{op}}(f,j) \simeq \varinjlim_{(f,j)\in i\downarrow J} \mathbb{Z}_{\mathcal{J}} \circ J^{i,\mathrm{op}}(f,j) \simeq \varinjlim_{(f,j)\in i\downarrow J} \mathbb{Z} \simeq \mathbb{Z}$$

in $D^-(\mathcal{A}b)$. The conclusion follows from Proposition 4.4.10 and Proposition 3.4.2. □

5.2 Cofinality and amplitude of derived projective limits

Definition 5.2.1. Let $\mathcal{I}$ be a small filtering category. By a result of Deligne (see [1, Proposition 8.1.6]), there is a cofinal functor

$$\Phi : I \longrightarrow \mathcal{I}$$

where I is a small filtering ordered set. Since any non empty set of cardinal numbers has a minimum, we may assume that I has the smallest possible cardinality. This cardinality will be called *the cofinality of $\mathcal{I}$*. We denote it by $\mathrm{cf}(\mathcal{I})$.

Notation 5.2.2. For any $k \in \mathbb{N}$, we denote by ω_k the $(k+1)$-th infinite cardinal number. For example, ω_0 is the cardinality of $\mathbb{N}$, ω_1 is the smallest cardinal number which is strictly greater than ω_0, and so on.

The following result is due to Goblot (see [4, Théorème 3.1]).

Proposition 5.2.3. *Let I be a filtering ordered set such that $\#I \leq \omega_k$ with $k < \omega_0$. Then, for any functor*

$$X : I^{\mathrm{op}} \longrightarrow \mathcal{A}b,$$

we have

$$H^n(\mathrm{R}\varprojlim_{\alpha\in I} X(\alpha)) = 0 \qquad \forall n \geq k+2.$$

Theorem 5.2.4. *Let $\mathcal{E}$ be a quasi-abelian category with exact products. Consider a functor*

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

where $\mathcal{I}$ is a small filtering category. If $\mathrm{cf}(\mathcal{I}) < \omega_k$ with $k < \omega_0$, then

$$LH^n(\mathrm{R}\varprojlim_{i\in\mathcal{I}} X(i)) = 0 \qquad \forall n \geq k+1.$$

Proof. Let $\Phi : I \longrightarrow \mathcal{I}$ be a cofinal functor where I is a filtering ordered set of cardinality $\mathrm{cf}(\mathcal{I})$. By Proposition 5.1.3, we have

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} X(i) \simeq \mathrm{R}\varprojlim_{i\in I} X\circ\Phi(i) \simeq R^{\cdot}(I, X\circ\Phi). \tag{*}$$

If $k = 0$, $\mathrm{cf}(\mathcal{I})$ is finite. Hence, $\mathrm{cf}(\mathcal{I}) = 1$ and the result is obvious.

Assume that $k > 0$. For any object Z of $\mathcal{E}$, we have

$$\begin{aligned}\mathrm{Hom}\,(Z, R^{\cdot}(I, X\circ\Phi)) &\simeq R^{\cdot}(I, \mathrm{Hom}\,(Z, X\circ\Phi))\\ &\simeq \mathrm{R}\varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{E}}(Z, X\circ\Phi(i)).\end{aligned}$$

Since $\#I = \mathrm{cf}(\mathcal{I}) \leq \omega_{k-1}$, by the preceding proposition, we get

$$H^n(\mathrm{R}\varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{E}}(Z, X\circ\Phi(i))) = 0 \qquad \forall n \geq k+1.$$

Then, for any object Z of $\mathcal{E}$,

$$H^n(\mathrm{Hom}\,(Z, R^{\cdot}(I, X\circ\Phi))) = 0 \qquad \forall n \geq k+1.$$

Hence, the complex $\mathrm{Hom}\,(Z, R^{\cdot}(I, X\circ\Phi))$ is exact in degree $n \geq k+1$ for any $Z \in \mathrm{Ob}(\mathcal{E})$. Therefore, by Remark 1.2.2, the complex $R^{\cdot}(I, X\circ\Phi)$ is strictly exact in degree $n \geq k+1$. Thanks to the isomorphisms (*), the complex $\mathrm{R}\varprojlim_{i\in\mathcal{I}} X(i)$ is strictly exact in degree $n \geq k+1$. $\square$

6 Pro-objects

In this section, we recall the definitions and results about pro-objects we need later (see [2] for more details).

6.1 Categories of pro-objects

Definition 6.1.1. Let $\mathcal{C}$ be an arbitrary category. Denote by

$$\mathcal{P}ro(\mathcal{C})$$

the category of *pro-objects* of $\mathcal{C}$. An object of $\mathcal{P}ro(\mathcal{C})$ is a functor

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

from a small filtering category $\mathcal{I}$ to $\mathcal{C}$. If

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C} \quad \text{and} \quad Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

are two pro-objects, then we set

$$\mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y) = \varprojlim_{j \in \mathcal{J}} [\varinjlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j))]$$

For any $i \in \mathcal{I}$ and any $j \in \mathcal{J}$, let

$$r_{ji} : \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j)) \longrightarrow \varinjlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j)),$$

$$q_j : \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y) \longrightarrow \varinjlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j))$$

be the canonical morphisms. We will say that a morphism

$$f_{ji} \in \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j))$$

represents an element

$$f \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y)$$

if

$$r_{ji}(f_{ji}) = q_j(f).$$

Clearly, if $f \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y)$ and $j \in \mathcal{J}$, there is $i \in \mathcal{I}$ and

$$f_{ji} \in \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j))$$

representing f.

Let
$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}, \quad Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C} \quad \text{and} \quad Z : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{C}$$
be three pro-objects. Consider
$$f \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y) \quad \text{and} \quad g \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(Y, Z).$$
We define $g \circ f$ as the element
$$h = (h_k)_{k \in \mathcal{K}} \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Z)$$
with
$$h_k = r_{ki}(g_{kj} \circ f_{ji})$$
where
$$f_{ji} \in \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j)) \quad \text{and} \quad g_{kj} \in \mathrm{Hom}_{\mathcal{C}}(Y(j), Z(k))$$
represent f and g respectively. This definition makes sense since for any $k \in \mathcal{K}$, one can check that h_k only depends of f and g.

Remark 6.1.2. Let $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$ and $Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be two pro-objects and let $f \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y)$. If
$$f_{ji} : X(i) \longrightarrow Y(j)$$
represents f then

(i) for any morphism $\alpha : i \longrightarrow i'$ of $\mathcal{I}$, the morphism
$$f_{ji} \circ X(\alpha) : X(i') \longrightarrow Y(j)$$
of $\mathcal{C}$ represents f,

(ii) for any morphism $\beta : j' \longrightarrow j$ of $\mathcal{J}$, the morphism
$$Y(\beta) \circ f_{ji} : X(i) \longrightarrow Y(j')$$
of $\mathcal{C}$ represents f.

Proposition 6.1.3. *Let $\mathcal{C}$ be an arbitrary category. The functor*
$$\text{“}\cdot\text{”} : \mathcal{C} \longrightarrow \mathcal{P}ro(\mathcal{C})$$
which associates to any object C of $\mathcal{C}$ the constant functor
$$C : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$$
where $\mathcal{I}$ is a one point category, is fully faithful. In particular, $\mathcal{C}$ may be identified with a full subcategory of $\mathcal{P}ro(\mathcal{C})$.

Definition 6.1.4. Let $\mathcal{C}$ be an arbitrary category and let $\mathcal{I}$ be a small filtering category. We define the functor

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”} : \mathcal{C}^{\mathcal{I}^{\text{op}}} \longrightarrow \mathcal{P}ro(\mathcal{C})$$

by setting

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, F(i) = F$$

for any functor $F : \mathcal{I}^{\text{op}} \longrightarrow \mathcal{C}$ and by setting

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, f(i) = [r_{ii}(f(i))]_{i\in\mathcal{I}}$$

for any morphism f of $\mathcal{C}^{\mathcal{I}^{\text{op}}}$.

Lemma 6.1.5. *Let $\mathcal{C}$ be a small category with filtering projective limits and let $\mathcal{I}$ be a small filtering category. For any object F of $\mathcal{C}^{\mathcal{I}^{\text{op}}}$, we have the isomorphism*

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, F(i) \simeq \varprojlim_{i\in\mathcal{I}} \text{“}F(i)\text{”}$$

of $\mathcal{P}ro(\mathcal{E})$.

In particular, if $\Phi : \mathcal{J} \longrightarrow \mathcal{I}$ is a cofinal functor from a small filtering category $\mathcal{J}$, we have the isomorphism

$$\text{“}\varprojlim_{j\in\mathcal{J}}\text{”}\, F(\Phi(j)) \simeq \text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, F(i)$$

in $\mathcal{P}ro(\mathcal{C})$.

Proposition 6.1.6. *If $\mathcal{I}$ is a small filtering category and if $\mathcal{C}$ is a category with finite projective limits (resp. finite inductive limits), then the functor*

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”} : \mathcal{C}^{\mathcal{I}^{\text{op}}} \longrightarrow \mathcal{P}ro(\mathcal{C})$$

commutes with finite projective limits (resp. finite inductive limits).

Definition 6.1.7. Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor between two arbitrary categories. We define a functor

$$\mathcal{P}ro(F) : \mathcal{P}ro(\mathcal{C}) \longrightarrow \mathcal{P}ro(\mathcal{C}')$$

by

$$\mathcal{P}ro(F)(X) = F \circ X$$

for any $X \in \mathrm{Ob}(\mathcal{P}ro(\mathcal{C}))$.

Consider two objects

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C} \quad \text{and} \quad Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

of $\mathcal{P}ro(\mathcal{C})$ and

$$f \in \operatorname{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, Y) = \varprojlim_{j \in \mathcal{J}} [\varinjlim_{i \in \mathcal{I}} \operatorname{Hom}_{\mathcal{C}}(X(i), Y(j))].$$

Then,

$$\mathcal{P}ro(F)(f) \in \operatorname{Hom}_{\mathcal{P}ro(\mathcal{C}')}(F \circ X, F \circ Y) = \varprojlim_{j \in \mathcal{J}} [\varinjlim_{i \in \mathcal{I}} \operatorname{Hom}_{\mathcal{C}'}(F(X(i)), F(Y(j)))]$$

is characterized by the relation

$$q_j(\mathcal{P}ro(F)(f)) = r_{ji}(F(f_{ji})) \qquad \forall j \in \mathcal{J}$$

where $f_{ji} \in \operatorname{Hom}_{\mathcal{C}}(X(i), Y(j))$ represents f.

In particular, for any functor

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

where $\mathcal{I}$ is a small filtering category, we have

$$\mathcal{P}ro(F)(\text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X(i)) \simeq \text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, F(X(i)).$$

6.2 Pro-representable functors

In this section, $\mathcal{C}$ denote an arbitrary category.

Proposition 6.2.1. *For any pro-object $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$, the functor*

$$\operatorname{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, \text{``}\cdot\text{''}) : \mathcal{C} \longrightarrow \mathcal{S}et$$

is isomorphic to the functor

$$\varinjlim_{i \in \mathcal{I}} h_{X(i)} : \mathcal{C} \longrightarrow \mathcal{S}et.$$

Proposition 6.2.2. *The functor*

$$H : \mathcal{P}ro(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathcal{S}et^{\mathcal{C}}$$

which associates to any object X of $\mathcal{P}ro(\mathcal{C})$ the functor

$$\operatorname{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, \text{``}\cdot\text{''}) : \mathcal{C} \longrightarrow \mathcal{S}et$$

and to any morphism $f : Y \longrightarrow X$ of pro-objects the morphism of functors

$$H(f) = \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(f, \text{“}\cdot\text{”}) : \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, \text{“}\cdot\text{”}) \longrightarrow \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(Y, \text{“}\cdot\text{”})$$

is fully faithful. In particular, the category $\mathcal{P}ro(\mathcal{C})^{\mathrm{op}}$ is isomorphic to a full subcategory of $\mathcal{S}et^{\mathcal{C}}$.

Definition 6.2.3. A *functor $F : \mathcal{C} \longrightarrow \mathcal{S}et$* is called *pro-representable* if it is in the essential image of H. In other words, F is pro-representable if there is $X \in \mathcal{P}ro(\mathcal{C})$ such that

$$F \simeq \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X, \text{“}\cdot\text{”}).$$

Proposition 6.2.4. *If $\mathcal{E}$ is a small category with finite projective limits, then a functor $F : \mathcal{E} \longrightarrow \mathcal{S}et$ is pro-representable if and only if it commutes with finite projective limits, i. e.,*

$$F(\varprojlim_{i \in \mathcal{I}} X(i)) = \varprojlim_{i \in \mathcal{I}} F(X(i))$$

where $X : \mathcal{I} \longrightarrow \mathcal{E}$ is a functor from a finite category.

6.3 Representation of diagrams of pro-objects

In this section, $\mathcal{C}$ denote an arbitrary category.

Definition 6.3.1. Let $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$ and $Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be two pro-objects and let $f : X \longrightarrow Y$ be a morphism of pro-objects. We denote by $\mathcal{M}_f$ the category whose objects are the morphisms which represent f. Let

$$\varphi : X(i) \longrightarrow Y(j) \qquad \text{and} \qquad \varphi' : X(i') \longrightarrow Y(j')$$

be two objects of $\mathcal{M}_f$. Then, a morphism

$$\varphi \longrightarrow \varphi'$$

is the data of a morphism $\alpha : i \longrightarrow i'$ of $\mathcal{I}$ and a morphism $\beta : j \longrightarrow j'$ of $\mathcal{J}$ such that the diagram

$$\begin{tikzcd} X(i') \arrow[r, "\varphi'"] \arrow[d, "X(\alpha)"'] & Y(j') \arrow[d, "Y(\beta)"] \\ X(i) \arrow[r, "\varphi"'] & Y(j) \end{tikzcd}$$

commutes.

Proposition 6.3.2. *Let $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$ and $Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be two pro-objects and let $f : X \longrightarrow Y$ be a morphism of pro-objects.*

(a) The category $\mathcal{M}_f$ is small and filtering. Moreover, the functors

$$p_{\mathcal{I}} : \mathcal{M}_f \longrightarrow \mathcal{I} \qquad \text{and} \qquad p_{\mathcal{J}} : \mathcal{M}_f \longrightarrow \mathcal{J}$$

which associate respectively to any morphism

$$\varphi : X(i) \longrightarrow Y(j)$$

representing f the object i of $\mathcal{I}$ and the object j of $\mathcal{J}$ are cofinal.

(b) The morphism

$$f' : X \circ p_{\mathcal{I}} \longrightarrow Y \circ p_{\mathcal{J}}$$

of $\mathcal{C}^{\mathcal{M}_f^{\mathrm{op}}}$ defined by

$$f'(\varphi) = \varphi$$

for any object $\varphi : X(i) \longrightarrow Y(j)$ of $\mathcal{M}_f$, represents f, i.e. the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\wr"] & Y \arrow[d, "\wr"] \\
\text{“}\varprojlim\limits_{\varphi \in \mathcal{M}_f}\text{”} (X \circ p_{\mathcal{I}})(\varphi) \arrow[r, "\text{“}\varprojlim\limits_{\varphi \in \mathcal{M}_f}\text{”} f'(\varphi)"'] & \text{“}\varprojlim\limits_{\varphi \in \mathcal{M}_f}\text{”} (Y \circ p_{\mathcal{J}})(\varphi)
\end{tikzcd}$$

of $\mathcal{P}ro(\mathcal{C})$ is commutative.

Proposition 6.3.3. *Let $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$ and $Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$ be two pro-objects and let $f : X \longrightarrow Y$ be a morphism of pro-objects. If there is a small filtering category $\mathcal{N}$, cofinal functors*

$$q_{\mathcal{I}} : \mathcal{N} \longrightarrow \mathcal{I} \quad \text{and} \quad q_{\mathcal{J}} : \mathcal{N} \longrightarrow \mathcal{J}$$

of $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ and a morphism of functors

$$g : X \circ q_{\mathcal{I}} \longrightarrow Y \circ q_{\mathcal{J}}$$

such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\wr"] & Y \arrow[d, "\wr"] \\
\text{“}\varprojlim\limits_{n \in \mathcal{N}}\text{”} (X \circ q_{\mathcal{I}})(n) \arrow[r, "\text{“}\varprojlim\limits_{n \in \mathcal{N}}\text{”} g(n)"'] & \text{“}\varprojlim\limits_{n \in \mathcal{N}}\text{”} (Y \circ q_{\mathcal{J}})(n)
\end{tikzcd}$$

is commutative in $\mathcal{P}ro(\mathcal{C})$*, then there is a cofinal functor*

$$r : \mathcal{N} \to \mathcal{M}_f$$

such that

$$q_{\mathcal{I}} = p_{\mathcal{I}} \circ r, \quad q_{\mathcal{J}} = p_{\mathcal{J}} \circ r$$

and

$$g = f'(r(\cdot)).$$

The basic results above have many useful variants. For example:

Proposition 6.3.4. *Let* $X : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$*,* $Y : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}$ *and* $Z : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}$ *be three pro-objects and let*

$$f : X \to Y \quad \text{and} \quad g : Y \to Z$$

be two morphisms of $\mathcal{P}ro(\mathcal{C})$*. Then, there is a small filtering category* $\mathcal{M}$*, cofinal functors*

$$p_{\mathcal{I}} : \mathcal{M} \to \mathcal{I}, \quad p_{\mathcal{J}} : \mathcal{M} \to \mathcal{J} \quad \text{and} \quad p_{\mathcal{K}} : \mathcal{M} \to \mathcal{K}$$

and morphisms

$$f' : X \circ p_{\mathcal{I}} \to Y \circ p_{\mathcal{J}} \quad \text{and} \quad g' : Y \circ p_{\mathcal{J}} \to Z \circ p_{\mathcal{K}}$$

such that f' *and* g' *represent* f *and* g*.*

Proposition 6.3.5. *Let* $X : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ *and* $Y : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}$ *be two pro-objects and let*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

be two morphisms of pro-objects. Then, there is a small filtering category $\mathcal{N}$ *and two morphisms of* $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$

$$\begin{tikzcd} X' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & Y' \end{tikzcd}$$

representing the morphisms f *and* g*.*

Proposition 6.3.6. *Let* $X : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$*,* $Y : \mathcal{J}^{\mathrm{op}} \to \mathcal{C}$ *and* $Z : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}$ *be three pro-objects and let*

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

be two morphisms of pro-objects. Then, there is a small filtering category $\mathcal{N}$ and two morphisms

$$\begin{tikzcd} X' \arrow[dr, "f'"] & \\ & Z' \\ Y' \arrow[ur, "g'"'] & \end{tikzcd}$$

of $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ representing the morphisms f and g.

Proposition 6.3.7. *If $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{C}$ and $Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{C}$ are two pro-objects, then there is a small filtering category $\mathcal{M}$ and two pro-objects*

$$X' : \mathcal{M}^{\mathrm{op}} \longrightarrow \mathcal{C} \quad \text{and} \quad Y' : \mathcal{M}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

such that

$$X \simeq X' \quad \text{and} \quad Y \simeq Y'.$$

Proposition 6.3.8. *Let $\mathcal{I}$ be a small filtering category. Two morphisms of $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$*

$$f : X \longrightarrow Y \quad \text{and} \quad g : X \longrightarrow Y$$

represent the same morphism of $\mathcal{P}ro(\mathcal{C})$ if and only if for any $i \in \mathcal{I}$, there is a morphism $\alpha : i \longrightarrow i'$ of $\mathcal{I}$ such that

$$f(i) \circ X(\alpha) = g(i) \circ X(\alpha).$$

6.4 Limits in categories of pro-objects

Proposition 6.4.1. *If the category $\mathcal{C}$ has finite projective limits then the category $\mathcal{P}ro(\mathcal{C})$ has finite projective limits .*

Proof. First, let us show that the category $\mathcal{P}ro(\mathcal{C})$ has finite products. If X and Y are two pro-objects, then by Proposition 6.3.7, there is a small filtering category $\mathcal{M}$ and two pro-objects $X' : \mathcal{M}^{\mathrm{op}} \longrightarrow \mathcal{C}$, $Y' : \mathcal{M}^{\mathrm{op}} \longrightarrow \mathcal{C}$ such that

$$X \simeq X' \quad \text{and} \quad Y \simeq Y'.$$

By Proposition 6.1.6, the product $X' \times Y'$ of $\mathcal{C}^{\mathcal{M}^{\mathrm{op}}}$ represents the product $X \times Y$ of $\mathcal{P}ro(\mathcal{C})$.

Next, let us show that the category $\mathcal{P}ro(\mathcal{C})$ has equalizers. If X and Y are two pro-objects, and if

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

are two morphisms of pro-objects, then by Proposition 6.3.5, there is a small filtering category $\mathcal{M}$ and two morphisms

$$X' \underset{g'}{\overset{f'}{\rightrightarrows}} Y'$$

of $\mathcal{C}^{\mathcal{M}^{\text{op}}}$ representing f and g. By Proposition 6.1.6, the equalizer $\text{eq}(f', g')$ of $\mathcal{C}^{\mathcal{M}^{\text{op}}}$ represents the equalizer $\text{eq}(f, g)$ of $\mathcal{P}ro(\mathcal{C})$.

Hence, the category $\mathcal{P}ro(\mathcal{C})$ has finite projective limits. □

Proposition 6.4.2. *If $\mathcal{C}$ is an arbitrary category then $\mathcal{P}ro(\mathcal{C})$ has filtering projective limits.*

Proof. Let

$$X : \mathcal{J}^{\text{op}} \longrightarrow \mathcal{P}ro(\mathcal{C})$$

be a functor where $\mathcal{J}$ is a small filtering category. Assume that for any $j \in \mathcal{J}$

$$X(j) : \mathcal{I}_j^{\text{op}} \longrightarrow \mathcal{C}$$

where $\mathcal{I}_j$ is a small filtering category.

Consider the category $\mathcal{K}$ whose objects are

$$\text{Ob}(\mathcal{K}) = \{(j, i) : j \in \mathcal{J},\ i \in \mathcal{I}_j\}.$$

A morphism

$$(j, i) \longrightarrow (j', i')$$

of $\mathcal{K}$ is the data of a morphism $\beta : j \longrightarrow j'$ of $\mathcal{J}$ and a morphism

$$f_{ii'} : X(j')(i') \longrightarrow X(j)(i)$$

of $\mathcal{C}$ which represents the morphism

$$X(\beta) : X(j') \longrightarrow X(j)$$

of $\mathcal{P}ro(\mathcal{C})$. One sees easily that $\mathcal{K}$ is filtering.

We define the pro-object

$$Y : \mathcal{K}^{\text{op}} \longrightarrow \mathcal{C}$$

by setting

$$Y(j, i) = X(j)(i)$$

for any object (j, i) of $\mathcal{K}$. If

$$u : (j, i) \longrightarrow (j', i')$$

is a morphism of $\mathcal{K}$ associated to the morphism $\beta : j \longrightarrow j'$ of $\mathcal{J}$ and to the morphism

$$f_{ii'} : X(j')(i') \to X(j)(i)$$

representing $X(\beta)$, then $Y(u)$ is defined by setting

$$Y(u) = f_{ii'}.$$

For any $j \in \mathcal{J}$, consider the morphism

$$x_j \in \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(Y, X(j)) = \varprojlim_{i \in \mathcal{I}_j} [\varinjlim_{(j',i') \in \mathcal{K}} \mathrm{Hom}_{\mathcal{C}}(Y(j',i'), X(j)(i))]$$

defined by

$$q_i^{(j)}(x_j) = r_{(j,i)(j,i)}(\mathrm{id}_{X(j)(i)})$$

for any $i \in \mathcal{I}_j$ where

$$q_i^{(j)} : \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(Y, X(j)) \longrightarrow \varinjlim_{(j',i') \in \mathcal{K}} \mathrm{Hom}_{\mathcal{C}}(Y(j',i'), X(j)(i))$$

and

$$r_{(j,i)(j',i')} : \mathrm{Hom}_{\mathcal{C}}(Y(j',i'), X(j)(i)) \longrightarrow \varinjlim_{(j',i') \in \mathcal{K}} \mathrm{Hom}_{\mathcal{C}}(Y(j',i'), X(j)(i))$$

are the canonical morphisms.

One can check that the pro-object

$$Y : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}$$

and the family of morphisms

$$x_j : Y \longrightarrow X(j)$$

form a projective limit of X. □

Lemma 6.4.3. *Let $\mathcal{C}$ be an arbitrary category and consider a functor*

$$X : \mathcal{J}^{\mathrm{op}} \to \mathcal{P}ro(\mathcal{C})$$

from a small filtering category $\mathcal{J}$. Then, for any object C of $\mathcal{C}$, we have

$$\varinjlim_{(j,i) \in \mathcal{K}} \mathrm{Hom}_{\mathcal{C}}(Y(j,i), C) \simeq \varinjlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}_j} \mathrm{Hom}_{\mathcal{C}}(X(j)(i), C)$$

where the categories $\mathcal{K}$, $\mathcal{I}_j$ and the functor

$$Y : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}$$

are defined in the proof of Proposition 6.4.2.

Proposition 6.4.4. *Let $\mathcal{C}$ be an arbitrary category and let*

$$X : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{P}ro(\mathcal{C})$$

be a functor from a small filtering category $\mathcal{J}$. Then, for any object C of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(\varprojlim_{j\in\mathcal{J}} X(j), \text{“}C\text{”}) \simeq \varinjlim_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X(j), \text{“}C\text{”}).$$

Proof. Assume that for any $j \in \mathcal{J}$

$$X(j) : \mathcal{I}_j^{\mathrm{op}} \longrightarrow \mathcal{C}$$

where $\mathcal{I}_j$ is a small filtering category. If

$$Y : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{C}$$

is the pro-object defined in Proposition 6.4.2, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(\varprojlim_{j\in\mathcal{J}} X(j), \text{“}C\text{”}) &\simeq \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(Y, \text{“}C\text{”}) \\
&\simeq \varinjlim_{(j,i)\in\mathcal{K}} \mathrm{Hom}_{\mathcal{C}}(Y(j,i), C) \\
&\simeq \varinjlim_{j\in\mathcal{J}} \varinjlim_{i\in\mathcal{I}_j} \mathrm{Hom}_{\mathcal{C}}(X(j)(i), C) \\
&\simeq \varinjlim_{j\in\mathcal{J}} \mathrm{Hom}_{\mathcal{P}ro(\mathcal{C})}(X(j), \text{“}C\text{”}).
\end{aligned}$$

□

7 Derived projective limits and pro-objects

7.1 Pro-objects of a quasi-abelian category

Proposition 7.1.1. *If $\mathcal{A}$ is an additive category then $\mathcal{P}ro(\mathcal{A})$ is canonically an additive category.*

Proof. Direct. □

Proposition 7.1.2. *Let $\mathcal{A}$ be an additive category with kernels and cokernels. If $f : X \longrightarrow Y$ is a morphism of $\mathcal{P}ro(\mathcal{A})$ represented by a morphism $f' : X' \longrightarrow Y'$ of $\mathcal{A}^{\mathcal{M}^{\mathrm{op}}}$, where $\mathcal{M}$ is a small filtering category, then* $\ker f'$ *(resp.* $\operatorname{coker} f'$*) represents* $\ker f$ *(resp.* $\operatorname{coker} f$*).*

Proof. This follows directly from the proof of Proposition 6.4.1. □

Remark 7.1.3. Let us recall that in an additive category with kernels and cokernels, f is a strict epimorphism if and only if f is a cokernel and f is a strict monomorphism if and only if f is a kernel.

Corollary 7.1.4. *Let $\mathcal{A}$ be an additive category with kernels and cokernels. A morphism $f : X \longrightarrow Y$ of $\mathcal{P}ro(\mathcal{A})$ is a strict epimorphism (resp. monomorphism) if and only if there is a small filtering category $\mathcal{M}$ and a strict epimorphism (resp. monomorphism) $f' : X' \longrightarrow Y'$ of $\mathcal{A}^{\mathcal{M}^{\mathrm{op}}}$ representing f.*

Proof. Consider a strict epimorphism $f : X \longrightarrow Y$ of $\mathcal{P}ro(\mathcal{A})$. Let $i : K \longrightarrow X$ be the kernel of f in $\mathcal{P}ro(\mathcal{A})$. We know that i can be represented by a morphism $i' : K' \longrightarrow X'$ of $\mathcal{A}^{\mathcal{M}^{\mathrm{op}}}$, where $\mathcal{M}$ is a small filtering category. If $f' : X' \longrightarrow Y'$ is the cokernel of i' in $\mathcal{A}^{\mathcal{M}^{\mathrm{op}}}$, then f' is a strict epimorphism. Moreover, by Proposition 7.1.2, f' represents $\operatorname{coker} i = f$.

Conversely, assume that the morphism $f : X \longrightarrow Y$ of $\mathcal{P}ro(\mathcal{A})$ is represented by a strict epimorphism $f' : X' \longrightarrow Y'$ of $\mathcal{A}^{\mathcal{M}^{\mathrm{op}}}$, where $\mathcal{M}$ is a small filtering category. Let $i' : K' \longrightarrow X'$ be the kernel of f'. Hence, $f' = \operatorname{coker} i'$. By Proposition 7.1.2,

$$\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, f'(m) : \text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, X'(m) \longrightarrow \text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, Y'(m)$$

is a cokernel of

$$\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, i'(m) : \text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, K'(m) \longrightarrow \text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, X'(m).$$

It follows that $\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, f'(m)$ is a strict epimorphism. Moreover, since f' represents f, there is a commutative diagram in $\mathcal{P}ro(\mathcal{A})$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\wr"] & Y \arrow[d, "\wr"] \\
\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, X'(m) \arrow[r, "\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, f'(m)"'] & \text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, Y'(m)
\end{tikzcd}$$

Therefore, f is a strict epimorphism in $\mathcal{P}ro(\mathcal{A})$. $\square$

Proposition 7.1.5. *If $\mathcal{E}$ is a quasi-abelian category then $\mathcal{P}ro(\mathcal{E})$ is a quasi-abelian category.*

Proof. By Proposition 7.1.1 and Proposition 7.1.2, the category $\mathcal{P}ro(\mathcal{E})$ is additive and has kernels and cokernels.

Let us show that if, in a cartesian square

$$\begin{tikzcd} X \arrow[r, "f"] & Z \\ T \arrow[u, "u"] \arrow[r, "v"'] & Y \arrow[u, "g"'] \end{tikzcd}$$

of $\mathcal{P}ro(\mathcal{E})$, f is a strict epimorphism then v is a strict epimorphism. Since f is a strict epimorphism, by Corollary 7.1.4, we can represent it by a strict epimorphism $f' : X' \to Z'$ of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$, where $\mathcal{M}$ is a small filtering category. Modifying $\mathcal{M}$ if necessary, by Proposition 6.3.6, we may assume that g is also represented by a morphism $g' : Y' \to Z'$ of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$. Since the category $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$ is quasi-abelian, we can form the cartesian square in $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$

$$\begin{tikzcd} X' \arrow[r, "f'"] & Z' \\ T' \arrow[u, "u'"] \arrow[r, "v'"'] & Y' \arrow[u, "g'"'] \end{tikzcd}$$

in which v' is a strict epimorphism of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$. By Proposition 6.1.6, the square

$$\begin{tikzcd} \text{“}\varprojlim\text{”}_{m\in\mathcal{M}} X'(m) \arrow[r, "\text{“}\varprojlim\text{”}_{m\in\mathcal{M}} f'(m)"] & \text{“}\varprojlim\text{”}_{m\in\mathcal{M}} Z'(m) \\ \text{“}\varprojlim\text{”}_{m\in\mathcal{M}} T'(m) \arrow[u, "\text{“}\varprojlim\text{”}_{m\in\mathcal{M}} u'(m)"] \arrow[r, "\text{“}\varprojlim\text{”}_{m\in\mathcal{M}} v'(m)"'] & \text{“}\varprojlim\text{”}_{m\in\mathcal{M}} Y'(m) \arrow[u, "\text{“}\varprojlim\text{”}_{m\in\mathcal{M}} g'(m)"'] \end{tikzcd}$$

is cartesian in $\mathcal{P}ro(\mathcal{E})$. Since f and g are isomorphic to

$$\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, f'(m) \qquad \text{and} \qquad \text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, g'(m),$$

$\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, T'(m)$ is isomorphic to T in $\mathcal{P}ro(\mathcal{E})$. Consequently, the morphisms u' and v' represent the morphisms u and v. By Corollary 7.1.4, v is a strict epimorphism.

Using the same kind of arguments, one can check that if, in a cocartesian square

$$\begin{tikzcd} Z \arrow[r, "v"] & T \\ X \arrow[u, "g"] \arrow[r, "f"'] & Y \arrow[u, "u"'] \end{tikzcd}$$

of $\mathcal{P}ro(\mathcal{E})$, f is a strict monomorphism then v is also a strict monomorphism. $\square$

Lemma 7.1.6. *If $\mathcal{E}$ is a quasi-abelian category with filtering projective limits, then $\mathcal{E}$ is complete.*

Proof. Since $\mathcal{E}$ has kernels and finite products, $\mathcal{E}$ has finite projective limits. Since $\mathcal{E}$ has also filtering projective limits, $\mathcal{E}$ is complete. □

Proposition 7.1.7. *If $\mathcal{E}$ is a quasi-abelian category, then $\mathcal{P}ro(\mathcal{E})$ is complete.*

Proof. We know that $\mathcal{P}ro(\mathcal{E})$ is quasi-abelian and has filtering projective limits. Then, by the preceding lemma, $\mathcal{P}ro(\mathcal{E})$ is complete. □

Proposition 7.1.8. *If $\mathcal{E}$ is a quasi-abelian category, then in $\mathcal{P}ro(\mathcal{E})$ products and filtering projective limits are exact.*

Proof. Consider a small family $(E_i)_{i\in I}$ of $\mathcal{P}ro(\mathcal{E})$. We know that

$$\prod_{i\in I} E_i \simeq \varprojlim_{J\in\mathcal{P}_f(I)} \prod_{j\in J} E_j$$

where $\mathcal{P}_f(I)$ is the set of the finite subsets of I ordered by inclusion. Since finite products are exact, it is sufficient to prove that filtering projective limits are exact in $\mathcal{P}ro(\mathcal{E})$. Since the filtering projective limits are kernel preserving, we only have to check that there are cokernel preserving.

Let $\mathcal{I}$ be a small filtering category and let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a costrictly exact sequence of $(\mathcal{P}ro(\mathcal{E}))^{\mathcal{I}^{\mathrm{op}}}$. Hence, for any $i\in\mathcal{I}$ and any object C of $\mathcal{E}$, the sequence

$$0 \longrightarrow \mathrm{Hom}\,(Z(i), \text{“}C\text{”}) \longrightarrow \mathrm{Hom}\,(Y(i), \text{“}C\text{”}) \longrightarrow \mathrm{Hom}\,(X(i), \text{“}C\text{”})$$

is exact in $\mathcal{A}b$. Since the functor $\varinjlim\limits_{i\in\mathcal{I}} : \mathcal{A}b^{\mathcal{I}} \longrightarrow \mathcal{A}b$ is exact, the sequence

$$0 \longrightarrow \varinjlim_{i\in\mathcal{I}} \mathrm{Hom}\,(Z(i), \text{“}C\text{”}) \longrightarrow \varinjlim_{i\in\mathcal{I}} \mathrm{Hom}\,(Y(i), \text{“}C\text{”}) \longrightarrow \varinjlim_{i\in\mathcal{I}} \mathrm{Hom}\,(X(i), \text{“}C\text{”})$$

is exact in $\mathcal{A}b$. Then, by 6.4.4, for any object C of $\mathcal{E}$, the sequence

$$0 \longrightarrow \mathrm{Hom}\,(\varprojlim_{i\in\mathcal{I}} Z(i), \text{“}C\text{”}) \longrightarrow \mathrm{Hom}\,(\varprojlim_{i\in\mathcal{I}} Y(i), \text{“}C\text{”}) \longrightarrow \mathrm{Hom}\,(\varprojlim_{i\in\mathcal{I}} X(i), \text{“}C\text{”})$$

is exact in $\mathcal{A}b$.

Now, consider a pro-object $E : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{E}$ of $\mathcal{E}$. By what precedes, for any $j \in \mathcal{J}$, the sequence

$$0 \longrightarrow \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} Z(i), \text{“}E(j)\text{”}) \longrightarrow \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} Y(i), \text{“}E(j)\text{”}) \longrightarrow \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} X(i), \text{“}E(j)\text{”})$$

is exact in $\mathcal{A}b$. Since the functor $\varprojlim_{j\in\mathcal{J}}$ is kernel preserving, the sequence

$$0 \longrightarrow \varprojlim_{j\in\mathcal{J}} \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} Z(i), \text{“}E(j)\text{”}) \longrightarrow \varprojlim_{j\in\mathcal{J}} \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} Y(i), \text{“}E(j)\text{”}) \longrightarrow \varprojlim_{j\in\mathcal{J}} \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} X(i), \text{“}E(j)\text{”})$$

is exact in $\mathcal{A}b$. Since we have successively

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\varprojlim_{i\in\mathcal{I}} X(i), E) &\simeq \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\varprojlim_{i\in\mathcal{I}} X(i), \text{“}\varprojlim_{j\in\mathcal{J}}\text{”}\, E(j)) \\
&\simeq \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\varprojlim_{i\in\mathcal{I}} X(i), \varprojlim_{j\in\mathcal{J}} \text{“}E(j)\text{”}) \\
&\simeq \varprojlim_{j\in\mathcal{J}} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\varprojlim_{i\in\mathcal{I}} X(i), \text{“}E(j)\text{”}),
\end{aligned}$$

the sequence

$$0 \longrightarrow \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} Z(i), E) \longrightarrow \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} Y(i), E) \longrightarrow \operatorname{Hom}(\varprojlim_{i\in\mathcal{I}} X(i), E)$$

is exact in $\mathcal{A}b$. It follows that the sequence

$$\varprojlim_{i\in\mathcal{I}} X(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} Y(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} Z(i) \longrightarrow 0$$

is costrictly exact in $\mathcal{P}ro(\mathcal{E})$. Hence, the functor $\varprojlim_{i\in\mathcal{I}}$ is exact. $\square$

Proposition 7.1.9. *If $\mathcal{E}$ is a quasi-abelian category, then for any object E of $\mathcal{E}$, the object “E” is cosmall, i.e.*

$$\operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\prod_{i\in I} X_i, \text{“}E\text{”}) \simeq \bigoplus_{i\in I} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(X_i, \text{“}E\text{”})$$

for any small family $(X_i)_{i\in I}$ of $\mathcal{P}ro(\mathcal{E})$.

Proof. For any object E of $\mathcal{E}$, we have successively

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\prod_{i\in I} X_i, \text{“}E\text{”}) &\simeq \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\varprojlim_{J\in\mathcal{P}_f(I)} \prod_{j\in J} X_j, \text{“}E\text{”}) \\
&\simeq \varinjlim_{J\in\mathcal{P}_f(I)} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\prod_{j\in J} X_j, \text{“}E\text{”}) \\
&\simeq \varinjlim_{J\in\mathcal{P}_f(I)} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\bigoplus_{j\in J} X_j, \text{“}E\text{”}) \\
&\simeq \varinjlim_{J\in\mathcal{P}_f(I)} \prod_{j\in J} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(X_j, \text{“}E\text{”}) \\
&\simeq \varinjlim_{J\in\mathcal{P}_f(I)} \bigoplus_{j\in J} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(X_j, \text{“}E\text{”}) \\
&\simeq \bigoplus_{i\in I} \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(X_i, \text{“}E\text{”}).
\end{aligned}$$

□

7.2 The functor L

Proposition 7.2.1. *Let $\mathcal{E}$ be a quasi-abelian category with products. For any object E of $\mathcal{E}$ and any pro-object $X : \mathcal{I}^{\mathrm{op}} \to \mathcal{E}$ we have*

$$\operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, X) \simeq \operatorname{Hom}_{\mathcal{E}}(E, \varprojlim_{i\in\mathcal{I}} X(i)).$$

Definition 7.2.2. Let $\mathcal{E}$ be a quasi-abelian category with products. We define the functor

$$\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \to \mathcal{E}$$

by setting

$$\mathrm{L}(X) = \varprojlim_{i\in\mathcal{I}} X(i)$$

for any pro-object $X : \mathcal{I}^{\mathrm{op}} \to \mathcal{E}$. By the preceding proposition, for any object E of $\mathcal{E}$, we have

$$\operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, X) \simeq \operatorname{Hom}_{\mathcal{E}}(E, \mathrm{L}(X)).$$

Hence, for any morphism $f : X \to Y$ of $\mathcal{P}ro(\mathcal{E})$, the theory of representable functors allows us to define

$$\mathrm{L}(f) : \mathrm{L}(X) \to \mathrm{L}(Y)$$

as the unique morphism making the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, X) \arrow[r, "\mathrm{Hom}\,(\text{“}E\text{”}, f)"] & \mathrm{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, Y) \\
\mathrm{Hom}_{\mathcal{E}}(E, \mathrm{L}(X)) \arrow[r, "\mathrm{Hom}\,(E, \mathrm{L}(f))"'] \arrow[u, "\wr"] & \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{L}(Y)) \arrow[u, "\wr"]
\end{tikzcd}$$

commutative.

Proposition 7.2.3. *Let $\mathcal{E}$ be a quasi-abelian category with products. The functor*

$$\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \longrightarrow \mathcal{E}$$

is left exact.

Proof. Consider a strictly exact sequence

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y$$

of $\mathcal{P}ro(\mathcal{E})$. Since for any object E of $\mathcal{E}$, the first sequence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, K) \arrow[r] & \mathrm{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, X) \arrow[r] & \mathrm{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}E\text{”}, Y) \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{L}(K)) \arrow[r] \arrow[u, "\wr"] & \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{L}(X)) \arrow[r] \arrow[u, "\wr"] & \mathrm{Hom}_{\mathcal{E}}(E, \mathrm{L}(Y)) \arrow[u, "\wr"]
\end{tikzcd}$$

is exact, the conclusion follows. □

7.3 Derivation of L

Lemma 7.3.1. *Let $\mathcal{E}$ be a quasi-abelian category. If I is an injective object of $\mathcal{E}$, then “I” is an injective object of $\mathcal{P}ro(\mathcal{E})$.*

Proof. Let I be an injective object of $\mathcal{E}$ and let $f : X \longrightarrow Y$ be a strict monomorphism of $\mathcal{P}ro(\mathcal{E})$. We know that f can be represented by a strict monomorphism $f' : X' \longrightarrow Y'$ of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$ where $\mathcal{M}$ is a small filtering category. Hence, for any object m of $\mathcal{M}$, the sequence

$$\mathrm{Hom}_{\mathcal{E}}(Y'(m), I) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X'(m), I) \longrightarrow 0$$

is exact. Since the functor $\varinjlim\limits_{m \in \mathcal{M}} : \mathcal{A}b^{\mathcal{M}^{\mathrm{op}}} \longrightarrow \mathcal{A}b$ is exact, the sequence

$$\varinjlim_{m \in \mathcal{M}} \mathrm{Hom}_{\mathcal{E}}(Y'(m), I) \longrightarrow \varinjlim_{m \in \mathcal{M}} \mathrm{Hom}_{\mathcal{E}}(X'(m), I) \longrightarrow 0$$

is exact. Moreover, we have

$$\begin{aligned}\varinjlim_{m\in\mathcal{M}} \operatorname{Hom}_{\mathcal{E}}(X'(m), I) &\simeq \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, X'(m), \text{“}I\text{”})\\ &\simeq \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(X, \text{“}I\text{”}).\end{aligned}$$

It follows that the sequence

$$\operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(Y, \text{“}I\text{”}) \longrightarrow \operatorname{Hom}_{\mathcal{P}ro(\mathcal{E})}(X, \text{“}I\text{”}) \longrightarrow 0$$

is exact and that “I” is an injective object of $\mathcal{P}ro(\mathcal{E})$. □

Proposition 7.3.2. *Let $\mathcal{E}$ be a quasi-abelian category. If $\mathcal{E}$ has enough injective objects, then $\mathcal{P}ro(\mathcal{E})$ has enough injective objects.*

Proof. Consider a pro-object $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$. Since $\mathcal{P}ro(\mathcal{E})$ is a quasi-abelian category, we know that there is a strictly exact sequence

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} \text{“}X(i)\text{”} \xrightarrow{\alpha} \prod_{i\in\mathcal{I}} \text{“}X(i)\text{”} \xrightarrow{\beta} \prod_{f\in\mathrm{Ar}(\mathcal{I})} \text{“}X(\mathrm{ext}(f))\text{”}$$

of $\mathcal{P}ro(\mathcal{E})$. Since

$$\varprojlim_{i\in\mathcal{I}} \text{“}X(i)\text{”} \simeq \text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, X(i) \simeq X,$$

we have a strict monomorphism

$$\alpha' : X \longrightarrow \prod_{i\in\mathcal{I}} \text{“}X(i)\text{”}$$

of $\mathcal{P}ro(\mathcal{E})$. Moreover, since $\mathcal{E}$ has enough injective objects, for any $i \in \mathcal{I}$, there is a strict monomorphism $X(i) \longrightarrow I(i)$ where $I(i)$ is injective in $\mathcal{E}$. It follows that for any $i \in \mathcal{I}$, “$X(i)$” $\longrightarrow$ “$I(i)$” is a strict monomorphism and by Lemma 7.3.1, “$I(i)$” is injective in $\mathcal{P}ro(\mathcal{E})$. Since the product of strict monomorphisms is a strict monomorphism,

$$\prod_{i\in\mathcal{I}} \text{“}X(i)\text{”} \longrightarrow \prod_{i\in\mathcal{I}} \text{“}I(i)\text{”}$$

is a strict monomorphism and $\prod_{i\in\mathcal{I}}$ “$I(i)$” is injective in $\mathcal{P}ro(\mathcal{E})$. Since the composition of strict monomorphisms is a strict monomorphism,

$$X \longrightarrow \prod_{i\in\mathcal{I}} \text{“}I(i)\text{”}$$

is a strict monomorphism. □

Corollary 7.3.3. *Let $\mathcal{E}$ be a quasi-abelian category with products. If $\mathcal{E}$ has enough injective objects, then the functor*

$$\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \longrightarrow \mathcal{E}$$

is right derivable.

Proposition 7.3.4. *Let $\mathcal{E}$ be a quasi-abelian category with enough injective objects. If the products are exact in $\mathcal{E}$, then for any family $(E_i)_{i\in I}$ of $\mathcal{E}$, the pro-object*

$$\prod_{i\in I} \text{“}E_i\text{”}$$

is acyclic for the functor $\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \longrightarrow \mathcal{E}$.

Proof. Consider a family $(E_i)_{i\in I}$ of $\mathcal{E}$. For any $i \in I$, let

$$0 \longrightarrow I_i^0 \longrightarrow I_i^1 \longrightarrow \cdots$$

be an injective resolution of E_i. Since the functor “$\cdot$” is exact and since “I” is injective if I is injective,

$$0 \longrightarrow \text{“}I_i^0\text{”} \longrightarrow \text{“}I_i^1\text{”} \longrightarrow \cdots$$

is an injective resolution of “E_i” in $\mathcal{P}ro(\mathcal{E})$. By Proposition 7.1.8, the products are exact in $\mathcal{P}ro(\mathcal{E})$. Then,

$$0 \longrightarrow \prod_{i\in I} \text{“}I_i^0\text{”} \longrightarrow \prod_{i\in I} \text{“}I_i^1\text{”} \longrightarrow \cdots$$

is an injective resolution of $\prod_{i\in I}$ “E_i”. It follows that $\mathrm{RL}(\prod_{i\in I}$ “E_i”$)$ is given by the complex

$$0 \longrightarrow \mathrm{L}(\prod_{i\in I} \text{“}I_i^0\text{”}) \longrightarrow \mathrm{L}(\prod_{i\in I} \text{“}I_i^1\text{”}) \longrightarrow \cdots$$

Since for $l \geq 0$, we have

$$\mathrm{L}(\prod_{i\in I} \text{“}I_i^l\text{”}) \simeq \prod_{i\in I} \mathrm{L}(\text{“}I_i^l\text{”}) \simeq \prod_{i\in I} I_i^l,$$

$\mathrm{RL}(\prod_{i\in I}$ “E_i”$)$ is isomorphic to the complex

$$0 \longrightarrow \prod_{i\in I} I_i^0 \longrightarrow \prod_{i\in I} I_i^1 \longrightarrow \cdots$$

Moreover, since the products are exact in $\mathcal{E}$, this complex is an injective resolution of $\prod_{i\in I} E_i$. Therefore, we have the following chain of isomorphisms in $D^+(\mathcal{P}ro(\mathcal{E}))$:

$$\mathrm{RL}(\prod_{i\in I} \text{“}E_i\text{”}) \simeq \prod_{i\in I} E_i \simeq \prod_{i\in I} \mathrm{L}(\text{“}E_i\text{”}) \simeq \mathrm{L}(\prod_{i\in I} \text{“}E_i\text{”}).$$

□

Definition 7.3.5. Let $\mathcal{E}$ be a quasi-abelian category with exact products. Let us define the functor

$$L^{\cdot} : \mathcal{P}ro(\mathcal{E}) \longrightarrow D^{+}(\mathcal{E}).$$

We set

$$L^{\cdot}(X) = R^{\cdot}(\mathcal{I}, X)$$

for any pro-object $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$. Let

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E} \quad \text{and} \quad Y : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

be two pro-objects and consider a morphism $f : X \longrightarrow Y$ of $\mathcal{P}ro(\mathcal{E})$. We know that there is a small filtering category $\mathcal{M}$ and cofinal functors

$$I : \mathcal{M} \longrightarrow \mathcal{I} \quad \text{and} \quad J : \mathcal{M} \longrightarrow \mathcal{J}$$

such that f is represented by a morphism

$$f' : X \circ I \longrightarrow Y \circ J$$

of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$. By Proposition 5.1.3, we have the canonical isomorphisms

$$R^{\cdot}(I, X) : R^{\cdot}(\mathcal{I}, X) \xrightarrow{\sim} R^{\cdot}(\mathcal{M}, I^{+}(X))$$

and

$$R^{\cdot}(J, Y) : R^{\cdot}(\mathcal{J}, Y) \xrightarrow{\sim} R^{\cdot}(\mathcal{M}, J^{+}(Y))$$

of $D^{+}(\mathcal{E})$. So, there is a unique morphism

$$R^{\cdot}(\mathcal{I}, X) \longrightarrow R^{\cdot}(\mathcal{J}, Y)$$

of $D^{+}(\mathcal{E})$ making the diagram

$$\begin{tikzcd} R^{\cdot}(\mathcal{M}, X \circ I) \arrow[r, "R^{\cdot}(\mathcal{M}, f')"] & R^{\cdot}(\mathcal{M}, Y \circ J) \\ R^{\cdot}(\mathcal{I}, X) \arrow[u, "R^{\cdot}(I,X)", "\wr"'] \arrow[r] & R^{\cdot}(\mathcal{J}, Y) \arrow[u, "R^{\cdot}(J,Y)"', "\wr"] \end{tikzcd}$$

commutative.

One can check that the morphism

$$R^{\cdot}(\mathcal{I}, X) \longrightarrow R^{\cdot}(\mathcal{J}, Y)$$

defined above does not depend on the choice of the representative of f. We denote it

$$L^{\cdot}(f).$$

Proposition 7.3.6. *Let $\mathcal{E}$ be a quasi-abelian category with exact products. The functor*

$$\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \longrightarrow \mathcal{E}$$

is right derivable and for any pro-object $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$ we have

$$\mathrm{RL}(X) \simeq L^{\cdot}(X)$$

in $D^{+}(\mathcal{E})$.

Proof. Consider the family

$$\mathcal{F} = \{X \in \mathrm{Ob}(\mathcal{P}ro(\mathcal{E})) : LH^{k}(L^{\cdot}(X)) = 0 \quad \text{if} \quad k > 0\}.$$

First, let us remark that if E is an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, where $\mathcal{I}$ is a small filtering category, then

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i) \in \mathcal{F}$$

if and only if E is $\varprojlim\limits_{i\in\mathcal{I}}$-acyclic. As a matter of fact, we have successively

$$\begin{aligned}
\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i) \in \mathcal{F} &\iff LH^{k}R^{\cdot}(\mathcal{I}, E) = 0 && \forall k > 0\\
&\iff LH^{k}\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) = 0 && \forall k > 0\\
&\iff \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \varprojlim_{i\in\mathcal{I}} E(i)
\end{aligned}$$

Next, let us prove that the family $\mathcal{F}$ is L-injective.

(i) Consider a pro-object $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$. We know that there is a strict monomorphism $X \longrightarrow \Pi(\mathrm{O}(X))$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and that $\Pi(\mathrm{O}(X))$ is $\varprojlim\limits_{i\in\mathcal{I}}$-acyclic. Since the functor

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{P}ro(\mathcal{E})$$

is exact, we have a strict monomorphism

$$X \longrightarrow \text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, \Pi(\mathrm{O}(X))(i)$$

of $Pro(\mathcal{E})$ and $\text{“}\varprojlim\limits_{i\in\mathcal{I}}\text{”}\, \Pi(\mathrm{O}(X))(i)$ belongs to $\mathcal{F}$.

(ii) Consider

$$X' : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}, \quad X : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathcal{E} \quad \text{and} \quad X'' : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

three pro-objects and a strictly exact sequence

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$$

of $\mathcal{P}ro(\mathcal{E})$ where X', $X \in \mathcal{F}$. Since f is a strict monomorphism, we know that it can be represented by a strict monomorphism $f' : Y' \longrightarrow Y$ of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$, where $\mathcal{M}$ is a small filtering category. If (Y'', g') is the cokernel of f', then the sequence

$$0 \longrightarrow Y' \xrightarrow{f'} Y \xrightarrow{g'} Y'' \longrightarrow 0 \tag{*}$$

of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$ is strictly exact. Since $\operatorname{coker} f'$ represents $\operatorname{coker} f$, we have

$$\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, Y''(m) \simeq X''.$$

Since X and X' belong to $\mathcal{F}$, for any $k > 0$, we have

$$LH^k L^{\cdot}(X') = LH^k L^{\cdot}(\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, Y'(m)) = 0$$

and

$$LH^k L^{\cdot}(X) = LH^k L^{\cdot}(\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, Y(m)) = 0.$$

It follows that Y' and Y are $\varprojlim_{i\in\mathcal{I}}$-acyclic. Since the sequence (*) is strictly exact, the object Y'' of $\mathcal{E}^{\mathcal{M}^{\mathrm{op}}}$ is $\varprojlim_{i\in\mathcal{I}}$-acyclic. Therefore, $\text{“}\varprojlim_{m\in\mathcal{M}}\text{”}\, Y''(m)$ belongs to $\mathcal{F}$. Hence X'' also belongs to $\mathcal{F}$.

Moreover, since the objects Y', Y and Y'' are $\varprojlim_{i\in\mathcal{I}}$-acyclic, the sequence

$$0 \longrightarrow \varprojlim_{m\in\mathcal{M}} Y'(m) \longrightarrow \varprojlim_{m\in\mathcal{M}} Y(m) \longrightarrow \varprojlim_{m\in\mathcal{M}} Y''(m) \longrightarrow 0$$

of $\mathcal{E}$ is strictly exact. Since the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim\limits_{m\in\mathcal{M}} Y'(m) \arrow[r] \arrow[d, "\wr"] & \varprojlim\limits_{m\in\mathcal{M}} Y(m) \arrow[r] \arrow[d, "\wr"] & \varprojlim\limits_{m\in\mathcal{M}} Y''(m) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathrm{L}(\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, Y'(m)) \arrow[r] \arrow[d, "\wr"] & \mathrm{L}(\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, Y(m)) \arrow[r] \arrow[d, "\wr"] & \mathrm{L}(\text{“}\varprojlim\limits_{m\in\mathcal{M}}\text{”}\, Y''(m)) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathrm{L}(X') \arrow[r] & \mathrm{L}(X) \arrow[r] & \mathrm{L}(X'') \arrow[r] & 0
\end{tikzcd}$$

commutes in $\mathcal{E}$, the sequence

$$0 \longrightarrow \mathrm{L}(X') \longrightarrow \mathrm{L}(X) \longrightarrow \mathrm{L}(X'') \longrightarrow 0$$

is also strictly exact.

Finally, consider a pro-object $X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$. We know that $R^{\cdot}(X)$ is a $\varprojlim\limits_{i\in\mathcal{I}}$-acyclic resolution of X. Then, for any $n \geq 0$,

$$\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, R^n(X)(i) \in \mathcal{F}.$$

Therefore, $\text{“}\varprojlim\limits_{i\in\mathcal{I}}\text{”}\, R^{\cdot}(X)(i)$ is an L-injective resolution of X and we have successively

$$\begin{aligned}\mathrm{RL}(X) &\simeq \mathrm{L}(\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, R^{\cdot}(X)(i))\\ &\simeq \varprojlim_{i\in\mathcal{I}} R^{\cdot}(X)(i)\\ &\simeq R^{\cdot}(\mathcal{I}, X)\\ &\simeq L^{\cdot}(X).\end{aligned}$$

□

Corollary 7.3.7. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $\mathcal{I}$ be a small filtering category. Then, we have*

$$\mathrm{RL} \circ \text{“}\varprojlim_{i\in\mathcal{I}}\text{”} \simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}}.$$

In particular,

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \mathrm{R}\varprojlim_{j\in\mathcal{J}} F(j)$$

if E and F are two essentially equivalent filtering projective systems (i.e. if the pro-objects associated to E and F are isomorphic).

Proof. For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have successively

$$\begin{aligned}\mathrm{RL}(\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i)) &\simeq L^{\cdot}(\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i))\\ &\simeq R^{\cdot}(\mathcal{I}, E)\\ &\simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i).\end{aligned}$$

□

Lemma 7.3.8. *Let $\mathcal{E}$ be a quasi-abelian category with exact products and let $E : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$ be a pro-object of $\mathcal{E}$. If for any object X of $\mathcal{E}$, $\mathcal{P}ro(h_X)(E)$ is* L*-acyclic, then E is* L*-acyclic.*

Proof. Consider $X \in \mathrm{Ob}(\mathcal{E})$. We have successively

$$\begin{aligned}\mathrm{RL}(\mathcal{P}ro(h_X)(E)) &\simeq \mathrm{RL}(\mathcal{P}ro(h_X)(\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i)))\\ &\simeq \mathrm{RL}(\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, h_X(E(i)))\\ &\simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}} h_X(E(i)).\end{aligned}$$

Since $\mathcal{P}ro(h_X)(E)$ is L-acyclic, we get

$$\begin{aligned}\mathrm{R}\varprojlim_{i\in\mathcal{I}} h_X(E(i)) &\simeq \mathrm{L}(\mathcal{P}ro(h_X)(E))\\ &\simeq \varprojlim_{i\in\mathcal{I}} \mathcal{P}ro(h_X)(E)(i)\\ &\simeq \varprojlim_{i\in\mathcal{I}} h_X(E(i)).\end{aligned}$$

It follows that $\mathrm{Hom}\,(X, E)$ is $\varprojlim_{i\in\mathcal{I}}$-acyclic. By Proposition 3.6.4, E is $\varprojlim_{i\in\mathcal{I}}$-acyclic. Therefore, we have

$$\mathrm{RL}(E) \simeq \mathrm{RL}(\text{“}\varprojlim_{i\in\mathcal{I}}\text{”}\, E(i)) \simeq \mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \simeq \varprojlim_{i\in\mathcal{I}} E(i) \simeq \mathrm{L}(E)$$

and E is L-acyclic. □

Proposition 7.3.9. *Let $\mathcal{E}$ be a quasi-abelian category with exact products. For any family $(E_i)_{i\in I}$ of $\mathcal{E}$, the pro-object $\prod_{i\in I}$ “E_i” is acyclic for*

$$\mathrm{L} : \mathcal{P}ro(\mathcal{E}) \longrightarrow \mathcal{E}.$$

Proof. By the preceding proposition, it is sufficient to show that for any object X of $\mathcal{E}$,

$$\mathcal{P}ro(h_X)(\prod_{i\in I} \text{“}E_i\text{”})$$

is L-acyclic. Consider an object X of $\mathcal{E}$. We get successively

$$\begin{aligned}\mathcal{P}ro(h_X)(\prod_{i\in I} \text{“}E_i\text{”}) &\simeq \mathcal{P}ro(h_X)(\varprojlim_{J\in\mathcal{P}_f(I)} (\prod_{j\in J} \text{“}E_j\text{”}))\\ &\simeq \mathcal{P}ro(h_X)(\varprojlim_{J\in\mathcal{P}_f(I)} (\text{“}\prod_{j\in J} E_j\text{”}))\end{aligned}$$

$$
\begin{aligned}
&\simeq \mathcal{P}ro(h_X)(\text{“}\varprojlim_{J\in\mathcal{P}_f(I)}\text{”}\,(\prod_{j\in J} E_j))\\
&\simeq \text{“}\varprojlim_{J\in\mathcal{P}_f(I)}\text{”}\,(h_X(\prod_{j\in J} E_j))\\
&\simeq \text{“}\varprojlim_{J\in\mathcal{P}_f(I)}\text{”}\,\mathrm{Hom}_{\mathcal{E}}(X,\prod_{j\in J} E_j)\\
&\simeq \text{“}\varprojlim_{J\in\mathcal{P}_f(I)}\text{”}\,\prod_{j\in J}\mathrm{Hom}_{\mathcal{E}}(X, E_j)\\
&\simeq \varprojlim_{j\in\mathcal{P}_f(I)}\,\text{“}\prod_{j\in J}\mathrm{Hom}_{\mathcal{E}}(X, E_j)\text{”}\\
&\simeq \varprojlim_{j\in\mathcal{P}_f(I)}\,\prod_{j\in J}\text{“}\mathrm{Hom}_{\mathcal{E}}(X, E_j)\text{”}\\
&\simeq \prod_{i\in I}\text{“}\mathrm{Hom}_{\mathcal{E}}(X, E_i)\text{”}\\
&\simeq \prod_{i\in I}\text{“}h_X(E_i)\text{”}.
\end{aligned}
$$

Since the category $\mathcal{A}b$ is abelian, has enough injective objects and has exact products, by Proposition 7.3.4,

$$\prod_{i\in I}\text{“}h_X(E_i)\text{”}$$

is L-acyclic. It follows that $\mathcal{P}ro(h_X)(\prod_{i\in I}\text{“}E_i\text{”})$ is L-acyclic. □

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Author's address: Laboratoire Analyse, Géométrie et Applications
UMR 7539
Université Paris 13
Avenue J.-B. Clément
93430 Villetaneuse
e-mail: `prosmans@math.univ-paris13.fr`