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# Derived Projective Limits of Topological Abelian Groups

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**Abstract**

In this paper, we prove that the category $\mathcal{TAb}$ of topological abelian groups is quasi-abelian. Using results about derived projective limits in quasi-abelian categories, we study exactness properties of the projective limit functor in $\mathcal{TAb}$. If $X$ is a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set, we give a necessary and sufficient condition for the derived projective limit of $X$ to be strict. We also characterize the countable projective systems of complete metrizable abelian groups which are $\varprojlim$-acyclic in $\mathcal{TAb}$.

## Contents



## 0 Introduction

In this paper, we prove that the category $\mathcal{TAb}$ of topological abelian groups is quasi-abelian in the sense of [6] (see also [4]). This allows us to use the results about

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derived projective limits in quasi-abelian categories obtained in [5] to study exactness properties of the projective limit functor for topological abelian groups. In particular, if $X$ is a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set $I$, we give a necessary and sufficient condition for the complex

$$\mathrm{R}\varprojlim_{i\in I} X_i$$

to be strict. When we assume moreover that $I$ is countable and each $X_i$ is metrizable and complete, we also give a necessary and sufficient acyclicity condition. This last result is related to theorems of Palamodov (cf. [2, 3]).

In the first section, we recall the definition of the category $\mathcal{TAb}$ of topological abelian groups and the form of kernels and cokernels in this category. This allows us to characterize the strict morphisms of $\mathcal{TAb}$ and to establish that this category is quasi-abelian.

The first part of Section 2 is devoted to a review of some of the results on derived projective limits in quasi-abelian categories established in [5]. More precisely, we recall that if $\mathcal{E}$ is a quasi-abelian category with exact products, the projective limit functor is right derivable and that its derived functor is computable by means of Roos complexes. We also recall that if $J : \mathcal{J} \longrightarrow \mathcal{I}$ is a cofinal functor between small filtering categories and if $E$ is a projective system indexed by $\mathcal{I}$, then the derived projective limits of $E$ and $E \circ J$ are isomorphic. In order to be able to apply these results to $\mathcal{TAb}$, we end this section by showing that products are exact in this category.

In the third section, we study strictness properties of the derived projective limit functor in $\mathcal{TAb}$. We establish that if $X$ is a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set, the differential $d^k$ of its Roos complex is strict for $k \geq 1$ and that $d^0$ is strict if and only if $X$ satisfies condition SC (i.e. if and only if for any $i \in I$ and any neighborhood $U$ of zero in $X_i$, there is $j \geq i$ such that

$$x_{i,k}(X_k) \subset q_i(\varprojlim_{i\in I} X_i) + U$$

for any $k \geq j$). As a corollary, we get that a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set is $\varprojlim$-acyclic in $\mathcal{TAb}$ if and only if it is $\varprojlim$-acyclic in the category of abelian groups and satisfies condition SC.

In the last section, we limit our study to countable projective systems of $\mathcal{TAb}$. First, we establish a slight generalization of the classical Mittag-Leffler theorem for countable projective limits of complete metric spaces. Using this result and results of Section 3, we give a necessary and sufficient condition for a countable projective system of complete metrizable abelian groups to be $\varprojlim$-acyclic in $\mathcal{TAb}$.

To conclude this introduction, I want to thank J.-P. Schneiders for pointing out the research direction followed in this paper and for the useful discussions we had during its preparation.

## 1 The category $\mathcal{TAb}$ of topological abelian groups

In this paper, by a *topological abelian group*, we mean an abelian group $M$ endowed with a topology such that the applications

$$+ : M \times M \longrightarrow M$$

and

$$- : M \longrightarrow M$$

are continuous.

Recall (see e.g. [1]) that if $M$ is a topological abelian group, then there is a basis of neighborhoods of zero $\mathcal{V}$ such that

(TAb1) $\forall\, V \in \mathcal{V}$, $V \ni 0$,

(TAb2) $\forall\, V \in \mathcal{V}$, $V = -V$,

(TAb3) $\forall\, V_1, V_2 \in \mathcal{V}$, $\exists\, V_3 \in \mathcal{V}$ such that $V_1 \cap V_2 \supset V_3$,

(TAb4) $\forall\, V \in \mathcal{V}$, $\exists\, U \in \mathcal{V}$ such that $U + U \subset V$.

Conversely, let $\mathcal{V}$ be a set of subsets of an abelian group $M$ satisfying (TAb1)–(TAb4). Then, the collection $\mathcal{T}$ of subsets $U$ of $M$ such that

$$\forall\, x \in U, \quad \exists\, V \in \mathcal{V} \text{ such that } x + V \subset U$$

is a topology of abelian group on $M$ for which $\mathcal{V}$ is a basis of neighborhoods of zero.

Let $M$ be a topological abelian group, let $N$ be a subgroup of $M$ and let $\mathcal{V}$ be a basis of neighborhoods of zero on $M$. The set

$$\mathcal{V}' = \{V \cap N : V \in \mathcal{V}\}$$

is clearly a basis of neighborhoods of zero for a topology of abelian group on $N$. We call the topology so defined on $N$ the *induced topology*.

Similarly, if $q : M \longrightarrow M/N$ denotes the canonical morphism, the set

$$\mathcal{V}' = \{q(V) : V \in \mathcal{V}\}$$

forms a basis of neighborhoods of zero for a topology of abelian group on $M/N$. The topology so defined on $M/N$ is called the *quotient topology*.

**Definition 1.1.** We denote by $\mathcal{TA}b$ the category whose objects are the topological abelian groups and whose morphisms are the continuous additive maps.

**Proposition 1.2.** *The category $\mathcal{TA}b$ has products. More precisely, let $(M_\alpha)_{\alpha\in A}$ be a family of topological abelian groups and let $\mathcal{V}_\alpha$ be a basis of neighborhoods of zero on $M_\alpha$ ($\forall \alpha \in A$). Then, the product of the family $(M_\alpha)_{\alpha\in A}$ in $\mathcal{TA}b$ is obtained by endowing the abelian group*

$$\prod_{\alpha\in A} M_\alpha = \{(m_\alpha)_{\alpha\in A} : m_\alpha \in M_\alpha \quad \forall\, \alpha \in A\}$$

*with the topology associated to the basis of neighborhoods of zero*

$$\mathcal{V} = \{\prod_{\alpha\in A} V_\alpha : V_\alpha = M_\alpha \text{ or } V_\alpha \in \mathcal{V}_\alpha,\ \{\alpha : V_\alpha \neq M_\alpha\} \text{ is finite}\,\}.$$

**Corollary 1.3.** *The category $\mathcal{TA}b$ is additive.*

**Proposition 1.4.** *The category $\mathcal{TA}b$ has kernels and cokernels. More precisely, let $u : M \longrightarrow N$ be a morphism of $\mathcal{TA}b$.*

*(i) The subgroup $u^{-1}(\{0\})$ of $M$ endowed with the induced topology together with the canonical monomorphism $i : u^{-1}(\{0\}) \longrightarrow M$ form a kernel of $u$.*

*(ii) The quotient group $N/u(M)$ endowed with the quotient topology together with the canonical epimorphism $q : N \longrightarrow N/u(M)$ form a cokernel of $u$.*

*(iii) The image of $u$ is the subgroup $u(M)$ of $N$ endowed with the induced topology.*

*(iv) The coimage of $u$ is the quotient group $M/u^{-1}(\{0\})$ endowed with the quotient topology.*

*Proof.* (i) Let $X$ be an object of $\mathcal{TA}b$ and let $v : X \longrightarrow M$ be a morphism of $\mathcal{TA}b$ such that $u \circ v = 0$. Since $v(X) \subset u^{-1}(\{0\})$, the application

$$v' : X \longrightarrow u^{-1}(\{0\}) \qquad x \mapsto v(x)$$

is well-defined. One sees easily that $v'$ is additive, continuous and makes the diagram

$$\begin{tikzcd}
u^{-1}(\{0\}) \arrow[r, "i"] & M \arrow[r, "u"] & N \\
& X \arrow[ul, "v'"] \arrow[u, "v"] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutative. Since $v'$ is the unique application satisfying these properties,

$$(u^{-1}(\{0\}), i)$$

is a kernel of $u$.

(ii) Let $X$ be an object of $\mathcal{TAb}$ and let $v : N \longrightarrow X$ be a morphism of $\mathcal{TAb}$ such that $v \circ u = 0$. The application

$$v' : N/u(M) \longrightarrow X \qquad [n]_{u(M)} \mapsto v(n)$$

is well-defined and additive. Let us show that $v'$ is continuous. Consider a neighborhood of zero $V$ in $X$. Since $v^{-1}(V)$ is a neighborhood of zero in $N$, $q(v^{-1}(V))$ is a neighborhood of zero in $N/u(M)$. Moreover, we have

$$v'^{-1}(V) \supset q(q^{-1}(v'^{-1}(V))) = q((v' \circ q)^{-1}(V)) = q(v^{-1}(V)).$$

It follows that $v'^{-1}(V)$ is a neighborhood of zero in $N/u(M)$ and that $v'$ is continuous. Of course, $v'$ makes the diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[dr, "0"'] & N \arrow[r, "q"] \arrow[d, "v"] & N/u(M) \arrow[dl, "v'"] \\ & X & \end{tikzcd}$$

commutative. Since $v'$ is the unique application having these properties,

$$(N/u(M), q)$$

is a cokernel of $u$.

(iii) and (iv) follow from (i) and (ii). □

**Proposition 1.5.** *A morphism $u : M \longrightarrow N$ of $\mathcal{TAb}$ is strict if and only if for any neighborhood of zero $V$ in $M$, there is a neighborhood of zero $V'$ in $N$ such that*

$$u(V) \supset u(M) \cap V'.$$

*Proof.* By definition, $u : M \longrightarrow N$ is strict if and only if the canonical morphism $\tilde{u} : \operatorname{coim} u \longrightarrow \operatorname{im} u$ is an isomorphism. This canonical morphism

$$\tilde{u} : M/u^{-1}(\{0\}) \longrightarrow u(M)$$

is defined by

$$\tilde{u}([m]_{u^{-1}(\{0\})}) = u(m) \qquad \forall\, m \in M.$$

One checks easily that $\tilde{u}$ is bijective. Moreover, $\tilde{u}$ is continuous. Hence, $u$ is strict if and only if $\tilde{u}^{-1}$ is continuous.

So, we have to show that

$$\tilde{u}^{-1} : u(M) \longrightarrow M/u^{-1}(\{0\}) \qquad u(m) \mapsto [m]_{u^{-1}(\{0\})}$$

is continuous if and only if for any neighborhood of zero $V$ in $M$, there is a neighborhood of zero $V'$ in $N$ such that

$$u(V) \supset u(M) \cap V'.$$

The condition is necessary. As a matter of fact, let $V$ be a neighborhood of zero in $M$. If $q' : M \longrightarrow M/u^{-1}(\{0\})$ is the canonical morphism, $q'(V)$ is a neighborhood of zero in $M/u^{-1}(\{0\})$. Since $\tilde{u}^{-1}$ is continuous,

$$(\tilde{u}^{-1})^{-1}(q'(V)) = \tilde{u}(q'(V)) = u(V)$$

is a neighborhood of zero in $u(M)$. Hence, there is a neighborhood of zero $V'$ in $N$ such that

$$u(V) \supset V' \cap u(M).$$

The condition is also sufficient. Let $W$ be a neighborhood of zero in $M/u^{-1}(\{0\})$. There is a neighborhood of zero $V$ in $M$ such that $W \supset q'(V)$. By hypothesis, there is a neighborhood of zero $V'$ in $N$ such that

$$u(V) \supset u(M) \cap V'.$$

Therefore, we have

$$(\tilde{u}^{-1})^{-1}(W) = \tilde{u}(W) \supset \tilde{u}(q'(V)) = u(V) \supset u(M) \cap V'.$$

Since $u(M) \cap V'$ is a neighborhood of zero in $u(M)$, $(\tilde{u}^{-1})^{-1}(W)$ is a neighborhood of zero in $u(M)$. Hence, $\tilde{u}^{-1}$ is continuous. $\square$

**Proposition 1.6.** *The category $\mathcal{TAb}$ is quasi-abelian.*

*Proof.* We know that $\mathcal{TAb}$ is additive and has kernels and cokernels.

(i) Consider a cartesian square

$$\begin{tikzcd} M_0 \arrow[r, "u"] & N_0 \\ M_1 \arrow[u, "f"] \arrow[r, "v"'] & N_1 \arrow[u, "g"'] \end{tikzcd}$$

where $u$ is a strict epimorphism and let us show that $v$ is a strict epimorphism. Recall that if we set

$$\alpha = \begin{pmatrix} u & -g \end{pmatrix} : M_0 \oplus N_1 \longrightarrow N_0,$$

then we may assume that

$$M_1 = \ker \alpha = \{(m_0, n_1) : u(m_0) = g(n_1)\}$$

and that
$$f = p_{M_0} \circ i_\alpha \quad \text{and} \quad v = p_{N_1} \circ i_\alpha$$
where $i_\alpha : \ker\alpha \longrightarrow M_0 \oplus N_1$ is the canonical monomorphism.

Of course, the morphism $v$ is surjective. Let us prove that it is strict. Consider a neighborhood of zero $V$ in $M_1 = \ker\alpha$. We may assume that
$$V = (V_0 \times V_1') \cap \ker\alpha$$
where $V_0$ is a neighborhood of zero in $M_0$ and $V_1'$ is a neighborhood of zero in $N_1$. Since $u$ is strict, by Proposition 1.5, there is a neighborhood of zero $V_0'$ in $N_0$ such that
$$u(V_0) \supset u(M_0) \cap V_0'.$$
Then, $V_1' \cap g^{-1}(V_0')$ is a neighborhood of zero in $N_1$. Since
$$v(V) \supset v(M_1) \cap V_1' \cap g^{-1}(V_0'),$$
by Proposition 1.5, $v$ is strict.

(ii) Consider a cocartesian square
$$\begin{tikzcd} M_1 \arrow[r, "v"] & N_1 \\ M_0 \arrow[u, "f"] \arrow[r, "u"'] & N_0 \arrow[u, "g"'] \end{tikzcd}$$
where $u$ is a strict monomorphism. Let us show that $v$ is a strict monomorphism. Recall that if we set
$$\alpha = \begin{pmatrix} f \\ -u \end{pmatrix} : M_0 \longrightarrow M_1 \oplus N_0,$$
then we may assume that
$$N_1 = \operatorname{coker}\alpha = (M_1 \oplus N_0)/\alpha(M_0),$$
$$v = q_\alpha \circ i_{M_1} \qquad \text{and} \qquad g = q_\alpha \circ i_{N_0}$$
where $q_\alpha : M_1 \oplus N_0 \longrightarrow (M_1 \oplus N_0)/\alpha(M_0)$ is the canonical epimorphism.

Clearly, the morphism $v$ is injective. Let us prove that it is strict. Consider a neighborhood of zero $V_1$ in $M_1$. We know that there is a neighborhood of zero $U_1$ in $M_1$ such that
$$U_1 + U_1 \subset V_1.$$
Since $u$ is strict, there is a neighborhood of zero $V_0'$ in $N_0$ such that
$$u(f^{-1}(U_1)) \supset u(M_0) \cap V_0'.$$

Moreover, $q_\alpha(U_1 \times V_0')$ is a neighborhood of zero in $N_1 = M_1 \oplus N_0/\alpha(M_0)$. One can check that

$$v(V_1) \supset v(M_1) \cap q_\alpha(U_1 \times V_0').$$

Hence, $v$ is strict. □

## 2 General results on derived projective limits in $\mathcal{TA}b$

Let $\mathcal{E}$ be a quasi-abelian category and let $\mathcal{I}$ be a small category. Recall that $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ denotes the quasi-abelian category of functors from $\mathcal{I}^{\mathrm{op}}$ to $\mathcal{E}$ (also called projective systems of $\mathcal{E}$ indexed by $\mathcal{I}$). For the reader convenience, we recall how to derive the projective limit functor

$$\varprojlim_{i \in \mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

if $\mathcal{E}$ is a quasi-abelian category with exact products (see [5] for more details).

Note that, hereafter, we will often denote by the same symbol a set and its associated discrete category.

**Definition 2.1.** Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. We define the functor

$$\Pi : \mathcal{E}^{\mathrm{Ob}(\mathcal{I})} \longrightarrow \mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$$

by setting

$$\Pi(S)(i) = \prod_{j \xrightarrow{\alpha} i} S(j)$$

for any functor $S : \mathrm{Ob}(\mathcal{I}) \longrightarrow \mathcal{E}$ and for any $i \in \mathcal{I}$. Let $i$ be an object of $\mathcal{I}$. For any morphism $\alpha : j \longrightarrow i$ of $\mathcal{I}$, we denote by

$$p_{j \xrightarrow{\alpha} i} : \Pi(S)(i) \longrightarrow S(j)$$

the canonical projection.

A *projective system*

$$E : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

is of *product type* if there is an object $S$ of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$ such that

$$E \simeq \Pi(S)$$

in $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

We denote by

$$\mathrm{O} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$$

the canonical functor.

**Proposition 2.2.** *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products.*

*(a) For any object $S$ of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, we have the isomorphism*

$$\varprojlim_{i\in\mathcal{I}} \Pi(S)(i) \simeq \prod_{i\in\mathcal{I}} S(i).$$

*(b) For any object $E$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, the morphism*

$$f : E \longrightarrow \Pi(\mathrm{O}(E))$$

*defined by*

$$p_{j\xrightarrow{\alpha} i} \circ f(i) = E(\alpha)$$

*for any object $i$ of $\mathcal{I}$ and any morphism $\alpha : j \longrightarrow i$ of $\mathcal{I}$ is a strict monomorphism.*

**Definition 2.3.** Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. We define the functor

$$R^{\cdot}(\mathcal{I}, \cdot) : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow C^{+}(\mathcal{E})$$

in the following way. For any functor $E : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$, we set

$$R^n(\mathcal{I}, E) = 0 \qquad \forall n < 0$$

and

$$R^n(\mathcal{I}, E) = \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} E(i_0) \qquad \forall n \geq 0,$$

where

$$i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n$$

is a chain of morphisms of $\mathcal{I}$. Denoting by

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} : R^n(\mathcal{I}, E) \longrightarrow E(i_0)$$

the canonical projection, we define the differential

$$d^n_{R^{\cdot}(\mathcal{I},E)} : R^n(\mathcal{I}, E) \longrightarrow R^{n+1}(\mathcal{I}, E)$$

by setting

$$\begin{aligned} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \circ d^n_{R^{\cdot}(\mathcal{I},E)} = {} & E(\alpha_1) \circ p_{i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \\ & + \sum_{l=1}^{n} (-1)^l p_{i_0 \xrightarrow{\alpha_1} \cdots i_{l-1} \xrightarrow{\alpha_{l+1}\circ\alpha_l} i_{l+1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \\ & + (-1)^{n+1} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n}. \end{aligned}$$

We call $R^{\cdot}(\mathcal{I}, E)$ the *Roos complex of* $E$.

**Notation 2.4.** Let $E$ be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. For any $i \in \mathcal{I}$, we denote by

$$q_i : \varprojlim_{i \in \mathcal{I}} E(i) \longrightarrow E(i)$$

the canonical morphism.

**Proposition 2.5.** *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. For any object $E$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is a canonical isomorphism*

$$\epsilon^0(\mathcal{I}, E) : \varprojlim_{i \in \mathcal{I}} E(i) \xrightarrow{\sim} \ker d^0_{R^{\cdot}(\mathcal{I},E)}$$

*defined by*

$$p_i \circ \epsilon^0(\mathcal{I}, E) = q_i \qquad \forall i \in \mathcal{I}.$$

**Definition 2.6.** Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. An object $E$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ is a *Roos-acyclic projective system* if the co-augmented complex

$$0 \longrightarrow \varprojlim_{i \in \mathcal{I}} E(i) \longrightarrow R^0(\mathcal{I}, E) \longrightarrow R^1(\mathcal{I}, E) \longrightarrow \cdots$$

is strictly exact.

**Proposition 2.7.** *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with products. For any object $S$ of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, there is a canonical homotopy equivalence*

$$\prod_{i \in \mathcal{I}} S(i) \to R^{\cdot}(\mathcal{I}, \Pi(S)).$$

*In particular, $\Pi(S)$ is a Roos-acyclic projective system.*

**Proposition 2.8.** *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-abelian category with exact products. Then, the family*

$$\mathcal{F} = \{E \in \mathrm{Ob}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}) : E \text{ is Roos-acyclic}\}$$

*is $\varprojlim_{i \in \mathcal{I}}$-injective. In particular, the functor*

$$\varprojlim_{i \in \mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{E}$$

*is right derivable and for any object $E$ of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have a canonical isomorphism*

$$\mathrm{R}\varprojlim_{i \in \mathcal{I}} E(i) \simeq R^{\cdot}(\mathcal{I}, E).$$

**Proposition 2.9.** *Let $J : \mathcal{J} \longrightarrow \mathcal{I}$ be a cofinal functor between small filtering categories and let $\mathcal{E}$ be a quasi-abelian category with exact products. For any object $E$ of $D^+(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$, the canonical morphism*

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \mathrm{R}\varprojlim_{j\in\mathcal{J}} E(J(j))$$

*is an isomorphism in $D^+(\mathcal{E})$.*

Recall that if $\mathcal{I}$ is a small filtering category, there is a small filtering ordered set $I$ and a cofinal functor $\Phi : I \longrightarrow \mathcal{I}$. Since any non empty set of cardinal numbers has a minimum, we may assume that $I$ has the smallest possible cardinality. We call this cardinality the cofinality of $\mathcal{I}$ and denote it $\mathrm{cf}(\mathcal{I})$.

Recall also that for $k \in \mathbb{N}$, $\omega_k$ denotes the $(k+1)$-th infinite cardinal number.

**Theorem 2.10.** *Let $\mathcal{E}$ be a quasi-abelian category with exact products. Consider a functor*

$$X : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathcal{E}$$

*where $\mathcal{I}$ is a small filtering category. If $\mathrm{cf}(\mathcal{I}) < \omega_k$ with $k \in \mathbb{N}$, then*

$$LH^n(\mathrm{R}\varprojlim_{i\in\mathcal{I}} X(i)) = 0 \qquad \forall n \geq k+1.$$

Since we know already that $\mathcal{TA}b$ is quasi-abelian, the following proposition will allow us to apply the preceding results to treat derived projective limits of topological abelian groups.

**Proposition 2.11.** *Products are exact in $\mathcal{TA}b$.*

*Proof.* Let $I$ be a small set. The functor

$$\prod_{i\in I} : \mathcal{TA}b^I \longrightarrow \mathcal{TA}b$$

being kernel preserving, it is sufficient to show that the product of strict epimorphisms is a strict epimorphism. Consider a family

$$u_i : M_i \longrightarrow N_i \qquad \forall\, i \in I$$

of strict epimorphisms. Of course, the application

$$\prod_{i\in I} u_i : \prod_{i\in I} M_i \rightarrow \prod_{i\in I} N_i$$

is surjective. Let us show that it is strict. Consider a neighborhood of zero $V$ in $\prod_{i\in I} M_i$. We may assume that

$$V = \prod_{i\in I} V_i$$

where $V_i$ is a neighborhood of zero in $M_i$ such that for

$$i \notin \{i_1, \cdots, i_J\}, \qquad (J \in \mathbb{N})$$

we have $V_i = M_i$. Since for any $i \in I$, $u_i$ is strict, there is a neighborhood of zero $V'_i$ in $N_i$ such that

$$u_i(V_i) \supset u_i(M_i) \cap V'_i.$$

For $i \notin \{i_1, \cdots, i_J\}$, we may assume that $V'_i = N_i$. Hence,

$$V' = \prod_{i\in I} V'_i$$

is a neighborhood of zero in $\prod_{i\in I} N_i$ and

$$\prod_{i\in I} u_i(V_i) \supset \prod_{i\in I} u_i(M_i) \cap \prod_{i\in I} V'_i.$$

By Proposition 1.5, $\prod_{i\in I} u_i$ is strict. $\square$

**Proposition 2.12.** *Let $\mathcal{I}$ be a small category. The functor*

$$\varprojlim_{i\in\mathcal{I}} : \mathcal{TA}b^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{TA}b$$

*is right derivable and for any object $M$ of $\mathcal{TA}b^{\mathcal{I}^{\mathrm{op}}}$, we have*

$$\mathrm{R}\varprojlim_{i\in\mathcal{I}} M(i) \simeq R^{\cdot}(\mathcal{I}, M)$$

*where $R^{\cdot}(\mathcal{I}, M)$ is the Roos complex of $M$.*

*Proof.* This follows from Proposition 2.8. $\square$

# 3 Strictness properties of derived projective limits in $\mathcal{TA}b$

Our aim in this section is to give a condition for the complex

$$\mathrm{R}\varprojlim_{i\in I} X_i$$

to be strict (i.e. to have strict differentials). Thanks to the following lemma, this is equivalent to give a condition in order that

$$LH^k(\mathrm{R}\varprojlim_{i\in I} X_i) \in \mathcal{TA}b.$$

**Lemma 3.1.** *Let $\mathcal{E}$ be a quasi-abelian category and let*

$$X^{\cdot} : \cdots X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \cdots$$

*be a complex of $\mathcal{E}$. Then,*

*(a) $LH^k(X^{\cdot}) \in \mathcal{E}$ if and only if the differential $d^{k-1}$ is strict;*

*(b) $LH^k(X^{\cdot}) = 0$ if and only if the sequence*

$$X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1}$$

*is strictly exact.*

**Definition 3.2.** Let $I$ be a filtering ordered set. We say that a projective system $X \in \mathcal{TAb}^{I^{\text{op}}}$ satisfies condition SC if for any $i \in I$ and any neighborhood $U$ of zero in $X_i$, there is $j \geq i$ such that

$$x_{i,k}(X_k) \subset q_i(\varprojlim_{i \in I} X_i) + U \qquad \forall\, k \geq j.$$

**Remark 3.3.** Let $\mathcal{I}$ be a small category and let $F : \mathcal{I}^{\text{op}} \longrightarrow \mathcal{TAb}$ be a functor. One can check easily that $\varprojlim_{i \in \mathcal{I}} F(i)$ is the abelian group

$$\{(f_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} F(i) : F(\alpha) f_{i'} = f_i \quad \forall\, \alpha : i \longrightarrow i' \text{ in } \mathcal{I}\}$$

endowed with the topology induced by that of $\prod_{i \in \mathcal{I}} F(i)$.

If moreover $\mathcal{I}$ is filtering, then for any neighborhood of zero $V$ in $\varprojlim_{i \in \mathcal{I}} F(i)$, there is $i \in \mathcal{I}$ and a neighborhood of zero $U_i$ in $F(i)$ such that

$$V \supset q_i^{-1}(U_i).$$

As a matter of fact, we know that $V$ contains a neighborhood of the form

$$(\prod_{i \in \mathcal{I}} W_i) \cap \varprojlim_{i \in \mathcal{I}} F(i)$$

where

$$W_{i_1}, \ \cdots \ , W_{i_k} \qquad (k \in \mathbb{N})$$

are neighborhoods of zero in $F(i_1), \ \cdots \ F(i_k)$ respectively and $W_i = F(i)$ if and only if $i \notin \{i_1, \ \cdots \ , i_k\}$. Hence, we have

$$V \supset (\prod_{i \in \mathcal{I}} W_i) \cap \varprojlim_{i \in \mathcal{I}} F(i) = \bigcap_{l=1}^{k} q_{i_l}^{-1}(W_{i_l}).$$

Since $\mathcal{I}$ is filtering, there is $i \in \mathcal{I}$ and there are morphisms

$$\alpha_{i_l} : i_l \longrightarrow i \qquad l = 1, \cdots, k$$

of $\mathcal{I}$. Since

$$F(\alpha_{i_l}) : F(i) \longrightarrow F(i_l) \qquad l = 1, \cdots, k$$

is continuous,

$$U_i = \bigcap_{l=1}^{k} (F(\alpha_{i_l}))^{-1}(W_{i_l})$$

is a neighborhood of zero in $F(i)$ and we see easily that

$$q_i^{-1}(U_i) \subset V.$$

**Theorem 3.4.** *Let $I$ be a filtering ordered set and let $X$ be an object of $\mathcal{TA}b^{I^{\mathrm{op}}}$. Then,*

$$LH^1(\mathrm{R}\varprojlim_{i \in I} X_i) \in \mathcal{TA}b$$

*if and only if $X$ satisfies condition SC.*

*In particular, the differential $d^0_{R^{\cdot}(I,X)}$ of the Roos complex of $X$ is strict if and only if $X$ satisfies condition SC.*

*Proof.* (a) Let us prove that the condition is sufficient.

We will decompose the argument in two steps.

(i) First, let us show that it is sufficient to prove that if

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

is a strictly exact sequence of $\mathcal{TA}b^{I^{\mathrm{op}}}$, then $\varprojlim_{i \in I} v_i$ is a strict morphism.

Let $X$ be an object of $\mathcal{TA}b^{I^{\mathrm{op}}}$. We know that there is a strict monomorphism

$$e : X \longrightarrow \Pi(\mathrm{O}(X)).$$

If $(Z, q)$ is the cokernel of $e$, then the sequence

$$0 \longrightarrow X \xrightarrow{e} \Pi(\mathrm{O}(X)) \xrightarrow{q} Z \longrightarrow 0$$

is strictly exact and it gives rise to the long exact sequence

$$0 \longrightarrow \varprojlim_{i \in I} X_i \xrightarrow{\varprojlim_{i \in I} e_i} \varprojlim_{i \in I} \Pi(\mathrm{O}(X))(i) \xrightarrow{\varprojlim_{i \in I} q_i} \varprojlim_{i \in I} Z_i \longrightarrow LH^1(\mathrm{R}\varprojlim_{i \in I} X_i) \longrightarrow 0 \qquad (*)$$

of $\mathcal{LH}(\mathcal{TAb})$ since $\Pi(\mathrm{O}(X))$ is $\varprojlim_{i\in I}$-acyclic. Set

$$f = \varprojlim_{i\in I} q_i$$

and let

$$J : \mathcal{TAb} \longrightarrow \mathcal{LH}(\mathcal{TAb})$$

be the canonical functor. Since $f$ is strict, the sequence

$$\varprojlim_{i\in I} \Pi(\mathrm{O}(X))(i) \xrightarrow{f} \varprojlim_{i\in I} Z_i \longrightarrow \operatorname{coker} f \longrightarrow 0$$

is strictly exact in $\mathcal{TAb}$. Hence, it gives rise to an exact sequence in $\mathcal{LH}(\mathcal{TAb})$. Therefore,

$$\begin{aligned} J(\operatorname{coker} f) &\simeq \operatorname{coker}(J(f)) \\ &\simeq LH^1(\mathrm{R}\varprojlim_{i\in I} X_i) \end{aligned}$$

since the sequence (*) is exact and we have

$$LH^1(\mathrm{R}\varprojlim_{i\in I} X_i) \in \mathcal{TAb}.$$

(ii) Let us prove that if

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

is a strictly exact sequence of $\mathcal{TAb}^{I^{\mathrm{op}}}$ such that $X$ satisfies condition SC, then $\varprojlim_{i\in I} v_i$ is strict. For this, it sufficient to show that for any neighborhood of zero $V$ in $\varprojlim_{i\in I} Y_i$, there is a neighborhood of zero $V'$ in $\varprojlim_{i\in I} Z_i$ such that

$$(\varprojlim_{i\in I} v_i)(V) \supset (\varprojlim_{i\in I} v_i)(\varprojlim_{i\in I} Y_i) \cap V'.$$

Let $V$ be a neighborhood of zero in $\varprojlim_{i\in I} Y_i$. By Remark 3.3, $V$ contains a neighborhood of the form

$$q_i^{-1}(U_i)$$

where $U_i$ is a neighborhood of zero in $Y_i$ for some $i \in I$.

Consequently, it is sufficient to show that for any $i \in I$ and for any neighborhood of zero $V_i$ in $Y_i$ there is a neighborhood of zero $V'$ in $\varprojlim_{i\in I} Z_i$ such that

$$(\varprojlim_{i\in I} v_i)(q_i^{-1}(V_i)) \supset (\varprojlim_{i\in I} v_i)(\varprojlim_{i\in I} Y_i) \cap V'.$$

Let $i \in I$ and let $V_i$ be a neighborhood of zero in $Y_i$. There is a neighborhood of zero $V'_i$ in $Y_i$ such that $V'_i + V'_i \subset V_i$. Set $U'_i = u_i^{-1}(V'_i)$. By hypothesis, there is $j \geq i$ such that

$$x_{i,k}(X_k) \subset q_i(\varprojlim_{i \in I} X_i) + U'_i \qquad \forall\, k \geq j.$$

If we set $V'_j = y_{i,j}^{-1}(V'_i)$, since $v_j$ is strict, there is a neighborhood of zero $W_j$ in $Z_j$ such that

$$v_j(Y_j) \cap W_j \subset v_j(V'_j).$$

Since $v_j$ is an epimorphism, we get

$$W_j \subset v_j(V'_j).$$

Moreover, since $q_j$ is continuous, $q_j^{-1}(W_j)$ is a neighborhood of zero in $\varprojlim_{i \in I} Z_i$. To conclude, let us show that

$$(\varprojlim_{i \in I} v_i)(\varprojlim_{i \in I} Y_i) \cap q_j^{-1}(W_j) \subset (\varprojlim_{i \in I} v_i)(q_i^{-1}(V_i)).$$

Consider

$$\gamma \in (\varprojlim_{i \in I} v_i)(\varprojlim_{i \in I} Y_i) \cap q_j^{-1}(W_j).$$

Hence,

$$q_j(\gamma) \in W_j$$

and there is $\beta \in \varprojlim_{i \in I} Y_i$ such that

$$(\varprojlim_{i \in I} v_i)(\beta) = \gamma.$$

It follows that

$$q_j(\gamma) = v_j(q_j(\beta)) \in W_j$$

and since

$$W_j \subset v_j(V'_j),$$

there is $\beta'_j \in V'_j$ such that

$$v_j(q_j(\beta)) = v_j(\beta'_j).$$

Hence, we have

$$q_j(\beta) - \beta'_j \in \ker v_j = \operatorname{im} u_j$$

and there is $\alpha_j \in X_j$ such that

$$q_j(\beta) - \beta'_j = u_j(\alpha_j).$$

Remark that

$$q_i(\beta) - y_{i,j}(\beta'_j) = q_i(\beta) - y_{i,j}(q_j(\beta) - u_j(\alpha_j)) = (u_i \circ x_{i,j})(\alpha_j).$$

Now, thanks to the relation

$$x_{i,j}(X_j) \subset q_i(\varprojlim_{i\in I} X_i) + U'_i,$$

there is $\alpha' \in \varprojlim_{i\in I} X_i$ such that

$$x_{i,j}(\alpha_j) - q_i(\alpha') \in U'_i.$$

Then, we have successively

$$q_i(\beta-(\varprojlim_{i\in I} u_i)(\alpha')) = y_{i,j}(\beta'_j)+u_i(x_{i,j}(\alpha_j))-u_i(q_i(\alpha')) = y_{i,j}(\beta'_j)+u_i(x_{i,j}(\alpha_j)-q_i(\alpha')).$$

Since

$$y_{i,j}(\beta'_j) \in y_{i,j}(V'_j) \subset V'_i$$

and

$$u_i(x_{i,j}(\alpha_j) - q_i(\alpha')) \in u_i(U'_i) \subset V'_i,$$

we get

$$q_i(\beta - \varprojlim_{i\in I} u_i(\alpha')) \in V_i.$$

Moreover, since

$$(\varprojlim_{i\in I} v_i)(\beta - (\varprojlim_{i\in I} u_i)(\alpha')) = (\varprojlim_{i\in I} v_i)(\beta) = \gamma$$

we have

$$\gamma \in \varprojlim_{i\in I} v_i(q_i^{-1}(V_i))$$

and the sufficiency of the condition is established.

(b) Let us prove the necessity of the condition. Let $i$ be an element of $I$ and let $U$ be a neighborhood of zero in $X_i$.

We know that

$$\mathrm{R}\varprojlim_{i\in I} X_i \simeq R^{\cdot}(I, X).$$

Since

$$LH^1(\mathrm{R}\varprojlim_{i\in I} X_i) \in \mathcal{TAb},$$

by Lemma 3.1,

$$d^0_{R^{\cdot}(I,X)} : \prod_{i\in I} X_i \to \prod_{j\le i} X_j$$

is a strict morphism. Therefore, there is a finite family of pairs $(j_k, i_k)_{k\in K}$ such that

$$j_k \leq i_k \qquad \forall\, k \in K$$

and there are neighborhoods of zero $V_{j_k,i_k}$ in $X_{j_k}$ such that

$$d^0_{R^\cdot(I,X)}(\prod_{i\in I} X_i) \cap \bigcap_{k\in K} p^{-1}_{j_k,i_k}(V_{j_k,i_k}) \subset d^0_{R^\cdot(I,X)}(p_i^{-1}(U)). \tag{*}$$

Since $I$ is filtering, there is $m \in I$ such that

$$i \leq m, \quad i_k \leq m, \quad j_k \leq m \qquad \forall\, k \in K.$$

Consider $n \geq m$ and $\beta_n \in X_n$. If we set

$$\beta_l = \begin{cases} x_{l,n}(\beta_n) & \text{if} \quad l \leq n \\ 0 & \text{otherwise} \end{cases}$$

then $\beta = (\beta_l)_{l\in I} \in \prod_{i\in I} X_i$ and for any $k \in K$, we get

$$p_{j_k,i_k} \circ d^0_{R^\cdot(I,X)}(\beta) = x_{j_k,i_k} \circ p_{i_k}(\beta) - p_{j_k}(\beta) = 0.$$

It follows that

$$d^0_{R^\cdot(I,X)}(\beta) \in \bigcap_{k\in K} p^{-1}_{j_k,i_k}(V_{j_k,i_k})$$

and thanks to the relation (*), there is $\beta' \in p_i^{-1}(U)$ such that

$$d^0_{R^\cdot(I,X)}(\beta) = d^0_{R^\cdot(I,X)}(\beta').$$

Hence,

$$\beta - \beta' \in \ker d^0_{R^\cdot(I,X)}.$$

Recall that $\ker d^0_{R^\cdot(I,X)} = \operatorname{im}(\epsilon^0(I,X))$, where $\epsilon^0(I,X)$ denotes the canonical augmentation of the Roos complex. Therefore, there is $\alpha \in \varprojlim_{i\in I} X_i$ such that

$$\beta - \beta' = \epsilon^0(I,X)(\alpha).$$

Since $i \leq n$, we have

$$x_{i,n}(\beta_n) - p_i(\beta') = \beta_i - p_i(\beta') = p_i(\beta - \beta') = (p_i \circ \epsilon^0(I,X))(\alpha) = q_i(\alpha).$$

Consequently,

$$x_{i,n}(\beta_n) = p_i(\beta') + q_i(\alpha)$$

and since $p_i(\beta') \in U$, we see that

$$x_{i,n}(\beta_n) \in U + q_i(\varprojlim_{i\in I} X_i).$$

The conclusion follows easily. □

**Theorem 3.5.** *Let $I$ be a filtering ordered set and let $X$ be an object of $\mathcal{TAb}^{I^{\mathrm{op}}}$. Then,*

$$LH^k(\mathrm{R}\varprojlim_{i\in I} X_i) \in \mathcal{TAb} \qquad \forall\, k \geq 2.$$

*In particular, the differential $d^k_{R^\cdot(I,X)}$ of the Roos complex of $X$ is strict for $k \geq 1$.*

*Proof.* We will decompose the argument in three steps.

(a) First, let us show that for any functor $S : \mathrm{Ob}(I) \longrightarrow \mathcal{TAb}$, the functor

$$\Pi(S) : I^{\mathrm{op}} \longrightarrow \mathcal{TAb}$$

verifies the condition SC. Consider $i \in I$ and $U$ a neighborhood of zero in

$$\Pi(S)(i) = \prod_{l \leq i} S_l.$$

If $k \geq i$, the morphism

$$p_{i,k} : \Pi(S)(k) \longrightarrow \Pi(S)(i)$$

is the canonical projection. Moreover, we know that

$$\varprojlim_{i\in I} \Pi(S)(i) \simeq \prod_{i\in I} S_i$$

and that

$$q_i : \varprojlim_{i\in I} \Pi(S)(i) \longrightarrow \Pi(S)(i)$$

is the canonical projection. It follows that

$$p_{i,k}(\Pi(S)(k)) = q_i(\varprojlim_{i\in I} \Pi(S)(i)) \subset q_i(\varprojlim_{i\in I} \Pi(S)(i)) + U.$$

(b) Next, consider an epimorphism $f : X \longrightarrow Y$ of $\mathcal{TAb}^{I^{\mathrm{op}}}$. Let us show that if $X$ verifies the condition SC, then $Y$ verifies the condition SC. Let $i \in I$ and let $V$ be a neighborhood of zero in $Y_i$. Since $f_i^{-1}(V)$ is a neighborhood of zero in $X_i$, there is $j \geq i$ such that

$$x_{i,k}(X_k) \subset q_i(\varprojlim_{i\in I} X_i) + f_i^{-1}(V) \qquad \forall\, k \geq j.$$

Consider $k \geq j$ and $y_k \in Y_k$. Since $f_k : X_k \longrightarrow Y_k$ is surjective, there is $x_k \in X_k$ such that $f_k(x_k) = y_k$. Then, there are $\alpha \in \varprojlim_{i\in I} X_i$ and $\beta \in f_i^{-1}(V)$ such that

$$x_{i,k}(x_k) = q_i(\alpha) + \beta.$$

Therefore, we get successively

$$y_{i,k}(y_k) = y_{i,k}(f_k(x_k)) = f_i(x_{i,k}(x_k)) = f_i(q_i(\alpha) + \beta) = q_i((\varprojlim_{i \in I} f_i)(\alpha)) + f_i(\beta).$$

It follows that

$$y_{i,k}(Y_k) \subset q_i(\varprojlim_{i \in I} Y_i) + V.$$

(c) Finally, let $X$ be an object of $\mathcal{TA}b^{I^{\mathrm{op}}}$. We know that there is a strict monomorphism

$$e : X \longrightarrow \Pi(\mathrm{O}(X)).$$

If $(Z, q)$ is the cokernel of $e$, the sequence

$$0 \longrightarrow X \xrightarrow{e} \Pi(\mathrm{O}(X)) \xrightarrow{q} Z \longrightarrow 0$$

is strictly exact and we get the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & LH^k(\mathrm{R}\varprojlim_{i \in I} \Pi(\mathrm{O}(X))(i)) \arrow[r] & LH^k(\mathrm{R}\varprojlim_{i \in I} Z_i) \arrow[dll, rounded corners] \\ LH^{k+1}(\mathrm{R}\varprojlim_{i \in I} X_i) \arrow[r] & LH^{k+1}(\mathrm{R}\varprojlim_{i \in I} \Pi(\mathrm{O}(X))(i)) \arrow[r] & \cdots \end{tikzcd}$$

Since $\Pi(\mathrm{O}(X))$ is $\varprojlim_{i \in I}$-acyclic, we have

$$LH^k(\mathrm{R}\varprojlim_{i \in I} \Pi(\mathrm{O}(X))(i)) = 0 \qquad \forall\, k \geq 1$$

and then

$$LH^k(\mathrm{R}\varprojlim_{i \in I} Z_i) \simeq LH^{k+1}(\mathrm{R}\varprojlim_{i \in I} X_i) \qquad \forall\, k \geq 1.$$

By (a), $\Pi(\mathrm{O}(X))$ verifies the condition SC and by (b), $Z$ verifies the condition SC. Then, by Theorem 3.4,

$$LH^1(\mathrm{R}\varprojlim_{i \in I} Z_i) \in \mathcal{TA}b$$

and the preceding isomorphism shows that

$$LH^2(\mathrm{R}\varprojlim_{i \in I} X_i) \in \mathcal{TA}b.$$

Reasoning by induction, we see easily that

$$LH^k(\mathrm{R}\varprojlim_{i \in I} X_i) \in \mathcal{TA}b \qquad \forall\, k \geq 2.$$

Finally, since

$$LH^k(\mathrm{R}\varprojlim_{i\in I} X_i) \simeq LH^k(R^{\cdot}(I,X)) \in \mathcal{TAb} \qquad \forall\, k \geq 2,$$

Lemma 3.1 shows that $d^k_{R^{\cdot}(I,X)}$ is strict for $k \geq 1$. $\square$

**Corollary 3.6.** *Let $\Phi : \mathcal{TAb} \longrightarrow \mathcal{Ab}$ be the forgetful functor which associates to any object $X$ of $\mathcal{TAb}$, the abelian group $X$. Let $I$ be a filtering ordered set. If $X$ is an object of $\mathcal{TAb}^{I^{\mathrm{op}}}$, then the following conditions are equivalent:*

*(i)* $\varprojlim_{i\in I} X_i \simeq \mathrm{R}\varprojlim_{i\in I} X_i$,

*(ii)* $\varprojlim_{i\in I} \Phi(X_i) \simeq \mathrm{R}\varprojlim_{i\in I} \Phi(X_i)$ *and $X$ satisfies condition SC.*

*Proof.* (i) $\Rightarrow$ (ii). Since $\varprojlim_{i\in I} X_i \simeq \mathrm{R}\varprojlim_{i\in I} X_i$, we have

$$LH^k(\mathrm{R}\varprojlim_{i\in I} X_i) = 0 \qquad \forall\, k \geq 1.$$

We know that

$$\mathrm{R}\varprojlim_{i\in I} X_i \simeq R^{\cdot}(I,X).$$

Hence, the sequence

$$R^{k-1}(I,X) \longrightarrow R^k(I,X) \longrightarrow R^{k+1}(I,X)$$

is strictly exact in $\mathcal{TAb}$ for $k \geq 1$. Therefore, this sequence is exact in $\mathcal{Ab}$. It follows that

$$LH^k(\mathrm{R}\varprojlim_{i\in I} \Phi(X_i)) = 0 \qquad \forall\, k \geq 1.$$

Moreover, the functor $\varprojlim_{i\in I} : \mathcal{Ab}^{I^{\mathrm{op}}} \longrightarrow \mathcal{Ab}$ being left exact, we have

$$LH^0(\mathrm{R}\varprojlim_{i\in I} \Phi(X_i)) \simeq \varprojlim_{i\in I} \Phi(X_i)$$

and we obtain

$$\varprojlim_{i\in I} \Phi(X_i) \simeq \mathrm{R}\varprojlim_{i\in I} \Phi(X_i).$$

Finally,

$$LH^1(\mathrm{R}\varprojlim_{i\in I} X_i) = 0 \in \mathcal{TAb}$$

and by Theorem 3.4, $X$ verifies the condition SC.

(ii) $\Rightarrow$ (i). By Theorem 3.4 and Theorem 3.5,

$$LH^k(\mathrm{R}\varprojlim_{i\in I} X_i) \in \mathcal{TAb} \qquad \forall\, k \geq 1.$$

Hence, $d^{k-1}_{R^\cdot(I,X)}$ is strict. Moreover, since

$$LH^k(\mathrm{R}\varprojlim_{i\in I} \Phi(X_i)) = 0 \qquad \forall\, k \geq 1,$$

we have

$$\ker d^k_{R^\cdot(I,X)} = \operatorname{im} d^{k-1}_{R^\cdot(I,X)}$$

in $\mathcal{Ab}$. Therefore, the sequence

$$R^{k-1}(I,X) \longrightarrow R^k(I,X) \longrightarrow R^{k+1}(I,X)$$

is strictly exact in $\mathcal{TAb}$ for $k \geq 1$ and

$$LH^k(\mathrm{R}\varprojlim_{i\in I} X_i) = 0 \qquad (k \geq 1).$$

Since

$$LH^0(\mathrm{R}\varprojlim_{i\in I} X_i) \simeq \varprojlim_{i\in I} X_i,$$

we obtain

$$\varprojlim_{i\in I} X_i \simeq \mathrm{R}\varprojlim_{i\in I} X_i.$$

□

## 4 An acyclicity condition for projective systems of $\mathcal{TAb}$

**Lemma 4.1.** *If $A$ is a countable filtering ordered set, there is a cofinal functor*

$$\alpha : \mathbb{N} \longrightarrow A.$$

*Proof.* Since $A$ is countable, there is a surjection $b : \mathbb{N} \longrightarrow A$. Since $A$ is filtering, we may find $\alpha(1) \in A$ such that

$$\alpha(1) \geq b(1).$$

In the same way, we may find $\alpha(2) \in A$ such that

$$\alpha(2) \geq b(2), \qquad \alpha(2) \geq \alpha(1).$$

By induction, we construct an increasing sequence $(\alpha(k))_{k\in\mathbb{N}}$ of $A$ such that

$$\alpha(k) \geq b(k) \qquad \forall\, k \in \mathbb{N}.$$

One checks easily that the functor

$$\alpha : \mathbb{N} \longrightarrow A$$

is cofinal. $\square$

**Remark 4.2.** Let $F$ be a subset of a metric space $E$. For any $\epsilon > 0$, we set

$$[F]_\epsilon = \{x \in E : d(x, F) < \epsilon\}.$$

Let us recall that if $f : E \longrightarrow F$ is an uniformly continuous application between two metric spaces, then for any $\epsilon > 0$, there is $\eta > 0$ such that

$$f([A]_\eta) \subset [f(A)]_\epsilon$$

for any subset $A$ of $E$.

**Proposition 4.3.** *Let $(X_a, x_{a,b})_{a\in A}$ be a filtering projective system of non-empty complete metric spaces and assume that $A$ has a countable cofinal subset. Assume that for $b \geq a$,*

$$x_{a,b} : X_b \longrightarrow X_a$$

*is uniformly continuous and that for any $a \in A$ and any $\epsilon > 0$, there is $b \geq a$ such that*

$$x_{a,b}(X_b) \subset [x_{a,c}(X_c)]_\epsilon \qquad \forall\, c \geq b.$$

*Then, for any $a \in A$ and any $\epsilon > 0$, there is $b \geq a$ such that*

$$x_{a,b}(X_b) \subset \left[ q_a(\varprojlim_{a\in A} X_a) \right]_\epsilon .$$

*In particular, $\varprojlim_{a\in A} X_a$ is not empty.*

*Proof.* We will decompose the proof in two steps.

(i) First, let us show that it is sufficient to prove the result for $A = \mathbb{N}$.

By the preceding lemma, there is a cofinal functor

$$\alpha : \mathbb{N} \longrightarrow A.$$

For any $k \in \mathbb{N}$, set

$$Y_k = X_{\alpha(k)}$$

and for $k \leq l$, set

$$y_{k,l} = x_{\alpha(k),\alpha(l)}.$$

(a) Let us prove that $(Y_k, y_{k,l})_{k\in\mathbb{N}}$ satisfies the same conditions as $(X_a, x_{a,b})_{a\in A}$. Of course, $(Y_k, y_{k,l})_{k\in\mathbb{N}}$ is a filtering countable projective system of complete metric spaces and for $k \leq l$,

$$y_{k,l} = x_{\alpha(k),\alpha(l)} : X_{\alpha(l)} \longrightarrow X_{\alpha(k)}$$

is uniformly continuous. Now, consider $k \in \mathbb{N}$ and $\epsilon > 0$. There is $b \geq \alpha(k)$ such that

$$x_{\alpha(k),b}(X_b) \subset \left[x_{\alpha(k),c}(X_c)\right]_\epsilon \qquad \forall\, c \geq b.$$

Since the functor $\alpha : \mathbb{N} \longrightarrow A$ is cofinal, there is $l \in \mathbb{N}$ such that $\alpha(l) \geq b$. Hence, $\alpha(l) \geq \alpha(k)$ and we have

$$y_{k,l}(Y_l) = x_{\alpha(k),b} \circ x_{b,\alpha(l)}(X_{\alpha(l)}) \subset x_{\alpha(k),b}(X_b).$$

If $m \geq l$, then $\alpha(m) \geq \alpha(l) \geq b$ and we get

$$y_{k,l}(Y_l) \subset x_{\alpha(k),b}(X_b) \subset \left[x_{\alpha(k),\alpha(m)}(X_{\alpha(m)})\right]_\epsilon \subset \left[y_{k,m}(Y_m)\right]_\epsilon.$$

(b) Now, let us show that if the result is true for $Y$, then it is for $X$.

Remark that since $\alpha$ is cofinal, we may assume that

$$\varprojlim_{k\in\mathbb{N}} Y_k = \varprojlim_{a\in A} X_a$$

and that the canonical morphism

$$q'_k : \varprojlim_{k\in\mathbb{N}} Y_k \longrightarrow Y_k$$

is $q_{\alpha(k)}$.

Consider $a \in A$ and $\epsilon > 0$. The functor $\alpha$ being cofinal, there is $k \in \mathbb{N}$ such that $\alpha(k) \geq a$. Since the application

$$x_{a,\alpha(k)} : X_{\alpha(k)} \longrightarrow X_a$$

is uniformly continuous, there is $\eta > 0$ such that

$$x_{a,\alpha(k)}(\left[q_{\alpha(k)}(\varprojlim_{a\in A} X_a)\right]_\eta) \subset \left[(x_{a,\alpha(k)} \circ q_{\alpha(k)})(\varprojlim_{a\in A} X_a)\right]_\epsilon.$$

Thanks to our assumption, there is $l \geq k$ such that

$$y_{k,l}(Y_l) \subset \left[q'_k(\varprojlim_{k\in\mathbb{N}} Y_k)\right]_\eta.$$

Hence, $\alpha(l) \geq \alpha(k) \geq a$ and we get

$$x_{a,\alpha(l)}(X_{\alpha(l)}) = x_{a,\alpha(k)}(y_{k,l}(Y_l)) \subset \left[ q_a(\varprojlim_{a \in A} X_a) \right]_{\epsilon}.$$

(ii) Next, let us prove the result for $A = \mathbb{N}$.

Consider $n \in \mathbb{N}$ and $\epsilon > 0$. Set $n_0 = n$ and choose $\epsilon_0 < \epsilon/2$.

(a) By induction, let us construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers and a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of strictly positive reals which converges to zero in such a way that

$$x_{n_k,n_{k+1}}(X_{n_{k+1}}) \subset [x_{n_k,n}(X_n)]_{\epsilon_k} \qquad \forall\, n \geq n_{k+1}$$

and

$$d(u,v) \leq \epsilon_k \implies d(x_{n_l,n_k}(u), x_{n_l,n_k}(v)) \leq 2^{l-k}\epsilon_l \qquad \forall\, l \leq k.$$

We have $n_0$ and $\epsilon_0$. By hypothesis, there is $n_1 > n_0$ such that

$$x_{n_0,n_1}(X_{n_1}) \subset [x_{n_0,n}(X_n)]_{\epsilon_0} \qquad \forall\, n \geq n_1$$

and since $x_{n_0,n_1} : X_{n_1} \longrightarrow X_{n_0}$ is uniformly continuous, there is $\epsilon_1 > 0$ such that

$$d(u,v) \leq \epsilon_1 \implies d(x_{n_0,n_1}(u), x_{n_0,n_1}(v)) \leq 2^{-1}\epsilon_0.$$

Suppose that we have constructed $n_i$ and $\epsilon_i$ for $i \leq k$ and let us construct $n_{k+1}$ and $\epsilon_{k+1}$. We know that there is $n_{k+1} > n_k$ such that

$$x_{n_k,n_{k+1}}(X_{n_{k+1}}) \subset [x_{n_k,n}(X_n)]_{\epsilon_k} \qquad \forall\, n \geq n_{k+1}.$$

For $l < k+1$, the application $x_{n_l,n_{k+1}} : X_{n_{k+1}} \longrightarrow X_{n_l}$ being uniformly continuous, there is $\eta_l > 0$ such that

$$d(u,v) \leq \eta_l \implies d(x_{n_l,n_{k+1}}(u), x_{n_l,n_{k+1}}(v)) \leq 2^{l-k-1}\epsilon_l.$$

If we set $\epsilon_{k+1} = \inf\{\eta_l : l < k+1\}$, then

$$d(u,v) \leq \epsilon_{k+1} \implies d(x_{n_l,n_{k+1}}(u), x_{n_l,n_{k+1}}(v)) \leq 2^{l-k-1}\epsilon_l \qquad \forall\, l \leq k+1.$$

(b) By induction, let us construct two sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}_0}$ such that

$$u_k = x_{n_k,n_{k+1}}(v_{k+1})$$

and

$$d(u_k, x_{n_k,n_{k+1}}(u_{k+1})) < \epsilon_k.$$

First, choose
$$u_0 \in x_{n_0,n_1}(X_{n_1}).$$
Hence,
$$u_0 = x_{n_0,n_1}(v_1), \qquad v_1 \in X_{n_1}.$$
Next, construct $u_1$ and $v_2$. By (ii)(a),
$$x_{n_0,n_1}(X_{n_1}) \subset [x_{n_0,n_2}(X_{n_2})]_{\epsilon_0}.$$
So, $u_0 \in [x_{n_0,n_2}(X_{n_2})]_{\epsilon_0}$ and there is $v_2 \in X_{n_2}$ such that
$$d(u_0, x_{n_0,n_2}(v_2)) < \epsilon_0.$$
Set $u_1 = x_{n_1,n_2}(v_2)$. Then, we have
$$d(u_0, x_{n_0,n_1}(u_1)) = d(u_0, x_{n_0,n_2}(v_2)) < \epsilon_0.$$

Finally, assume that we have constructed $u_0, \cdots, u_k$ and $v_1, \cdots, v_{k+1}$ and let us construct $u_{k+1}$ and $v_{k+2}$. We know that
$$u_k = x_{n_k,n_{k+1}}(v_{k+1})$$
and that
$$x_{n_k,n_{k+1}}(X_{n_{k+1}}) \subset \left[x_{n_k,n_{k+2}}(X_{n_{k+2}})\right]_{\epsilon_k}.$$
Then, there is $v_{k+2} \in X_{n_{k+2}}$ such that
$$d(u_k, x_{n_k,n_{k+2}}(v_{k+2})) < \epsilon_k.$$
If we set $u_{k+1} = x_{n_{k+1},n_{k+2}}(v_{k+2})$, then
$$d(u_k, x_{n_k,n_{k+1}}(u_{k+1})) = d(u_k, x_{n_k,n_{k+2}}(v_{k+2})) < \epsilon_k.$$

(c) Fix $l \in \mathbb{N}$. For $k \geq l$, set
$$w_k^l = x_{n_l,n_k}(u_k).$$
We get
$$d(w_k^l, w_{k+1}^l) = d(x_{n_l,n_k}(u_k), x_{n_l,n_{k+1}}(u_{k+1})) = d(x_{n_l,n_k}(u_k), x_{n_l,n_k}(x_{n_k,n_{k+1}}(u_{k+1}))).$$
By (ii)(b),
$$d(u_k, x_{n_k,n_{k+1}}(u_{k+1})) < \epsilon_k$$

and by (ii)(a),

$$d(w_k^l, w_{k+1}^l) \leq 2^{l-k}\epsilon_l.$$

So, for $q > p \geq l$, we have

$$d(w_p^l, w_q^l) \leq \sum_{k=p}^{q-1} d(w_k^l, w_{k+1}^l) \leq \sum_{k=p}^{q-1} 2^{l-k}\epsilon_l.$$

Hence, $(w_k^l)_{k\geq l}$ is a Cauchy sequence in $X_{n_l}$ and since $X_{n_l}$ is complete, this sequence converges. Denote $w^l$ its limit. We get successively

$$x_{n_l,n_{l+1}}(w^{l+1}) = \lim_{k\longrightarrow+\infty} x_{n_l,n_{l+1}}(w_k^{l+1}) = \lim_{k\longrightarrow+\infty} x_{n_l,n_k}(u_k) = w^l.$$

It follows that $(w^l)_{l\in\mathbb{N}} \in \varprojlim_{l\in\mathbb{N}} X_{n_l}$. Since the sequence $(n_l)_{l\in\mathbb{N}}$ is strictly increasing, the map

$$l \mapsto n_l$$

is cofinal and

$$\varprojlim_{l\in\mathbb{N}} X_{n_l} \xrightarrow{\sim} \varprojlim_{n\in\mathbb{N}} X_n.$$

Denote by $w'$ the image of $(w^l)_{l\in\mathbb{N}}$ by this isomorphism. For any $l \in \mathbb{N}$,

$$w^l = q_{n_l}(w').$$

Since for $q > p \geq l$,

$$d(w_p^l, w_q^l) \leq \sum_{k=p}^{q-1} 2^{l-k}\epsilon_l,$$

we have

$$d(w_0^0, w^0) \leq \sum_{k=0}^{\infty} 2^{-k}\epsilon_0 = 2\epsilon_0 < \epsilon.$$

Since $w_0^0 = x_{n_0,n_0}(u_0) = u_0$, we obtain

$$d(u_0, q_{n_0}(w')) = d(w_0^0, w^0) < \epsilon.$$

It follows that

$$u_0 \in \left[ q_{n_0}(\varprojlim_{n\in\mathbb{N}} X_n) \right]_\epsilon .$$

Since $u_0$ is an arbitrary element of $x_{n_0,n_1}(X_{n_1})$, we have

$$x_{n_0,n_1}(X_{n_1}) \subset \left[ q_{n_0}(\varprojlim_{n\in\mathbb{N}} X_n) \right]_\epsilon .$$

Recall that $n_0 = n$. Hence, we have found $n_1 \geq n$ such that

$$x_{n,n_1}(X_{n_1}) \subset \left[ q_n(\varprojlim_{n\in\mathbb{N}} X_n) \right]_\epsilon .$$

$\square$

**Remark 4.4.** Recall that a *topological abelian group* is *metrizable* if its topology may be defined by a metric and that the following conditions are equivalent:

(a) $M$ is metrizable,

(b) there is a countable basis of neighborhoods of zero $\mathcal{V}$ such that

$$\bigcap_{V\in\mathcal{V}} V = \{0\},$$

(c) there is an application $\|\cdot\| : M \longrightarrow [0, +\infty[$ such that

   (1) $\|-x\| = \|x\|$,
   (2) $\|x + y\| \leq \|x\| + \|y\|$,
   (3) $\|x\| = 0 \implies x = 0$,
   (4) $\{B(\epsilon) = \{x \in M : \|x\| < \epsilon\} : \epsilon > 0\}$ is a basis of neighborhoods of zero.

Note that in case (c), the metric of $M$ can be defined by

$$d(x, y) = \|x - y\| .$$

Conversely, in case (a), the application $\|\cdot\| : M \longrightarrow [0, +\infty[$ can be defined by

$$\|m\| = d(m, 0) \qquad \forall\, m \in M.$$

Of course, a metrizable topological abelian group is separated.

**Lemma 4.5.** *Let*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

*be an exact sequence of filtering projective systems of topological abelian groups indexed by $A$. Assume that $A$ has a countable cofinal subset. Assume moreover that for any $a \in A$, $X_a$ is metrizable and complete and that for any neighborhood of zero $V$ in $X_a$, there is $b \geq a$ such that*

$$x_{a,b}(X_b) \subset V + x_{a,c}(X_c) \qquad \forall\, c \geq b.$$

*Then, the sequence*

$$0 \longrightarrow \varprojlim_{a\in A} X_a \xrightarrow{\varprojlim_{a\in A} u_a} \varprojlim_{a\in A} Y_a \xrightarrow{\varprojlim_{a\in A} v_a} \varprojlim_{a\in A} Z_a \longrightarrow 0$$

*is exact in* $\mathcal{A}b$.

*Proof.* Since the functor $\varprojlim_{a\in A}$ is left exact, it is sufficient to show that

$$\varprojlim_{a\in A} v_a : \varprojlim_{a\in A} Y_a \longrightarrow \varprojlim_{a\in A} Z_a$$

is surjective.

Consider $z = (z_a)_{a\in A} \in \varprojlim_{a\in A} Z_a$. For any $a \in A$, set

$$M_a = \{m_a \in Y_a : v_a(m_a) = z_a\}.$$

Since $v_a$ is surjective, $M_a \neq \emptyset$. Choose $m_a^0 \in M_a$ and let us prove that the application

$$f_a : X_a \longrightarrow M_a$$

defined by

$$f_a(x_a) = u_a(x_a) + m_a^0, \qquad x_a \in X_a$$

is bijective. Of course, $f_a$ is injective. Consider $m_a \in M_a$. Since

$$v_a(m_a - m_a^0) = v_a(m_a) - v_a(m_a^0) = z_a - z_a = 0$$

and since $\operatorname{im} u_a = \ker v_a$, there is $x_a \in X_a$ such that

$$u_a(x_a) = m_a - m_a^0.$$

Therefore, $m_a = f_a(x_a)$ and $f_a$ is surjective.

For $b \geq a$, we have

$$v_a(y_{a,b}(m_b^0) - m_a^0) = z_{a,b}(v_b(m_b^0)) - z_a = z_{a,b}(z_b) - z_a = z_a - z_a = 0.$$

So, there is a unique $x_a^b \in X_a$ such that

$$u_a(x_a^b) = y_{a,b}(m_b^0) - m_a^0.$$

For $b \geq a$, consider the application

$$x'_{a,b} : X_b \longrightarrow X_a$$

defined by

$$x'_{a,b}(x_b) = x_{a,b}(x_b) + x_a^b, \qquad x_b \in X_b.$$

The diagram

$$\begin{array}{ccc} X_b & \xrightarrow{f_b} & M_b \\ {\scriptstyle x'_{a,b}}\downarrow & & \downarrow{\scriptstyle y_{a,b}} \\ X_a & \xrightarrow[f_a]{} & M_a \end{array}$$

is clearly commutative. Therefore, for $c \geq b \geq a$, we have

$$x'_{a,b} \circ x'_{b,c} = f_a^{-1} \circ y_{a,b} \circ y_{b,c} \circ f_c = x'_{a,c}.$$

Since $x_{a,b}$ is additive and continuous, $x_{a,b}$ is uniformly continuous. Hence, $x'_{a,b}$ is also uniformly continuous and we may consider $(X_a, x'_{a,b})_{a\in A}$ as a filtering projective system of complete metric spaces. We may also assume that the metric of $X_a$ is associated to an application

$$\|\cdot\|_a : X_a \longrightarrow [0, +\infty[$$

satisfying the conditions in part (c) of Remark 4.4.

Now, consider $a \in A$ and $\epsilon > 0$. We know that

$$B(\epsilon) = \{x \in X_a : \|x\|_a < \epsilon\}$$

is a neighborhood of zero in $X_a$. By hypothesis, there is $b \geq a$ such that

$$x_{a,b}(X_b) \subset B(\epsilon) + x_{a,c}(X_c) \qquad c \geq b.$$

Remark that for $c \geq b$ and for any $x_c \in X_c$, we have

$$\begin{aligned} x'_{a,b}(x'_{b,c}(x_c)) &= x'_{a,b}(x_{b,c}(x_c) + x_b^c) \\ &= x_{a,b}(x_{b,c}(x_c)) + x_{a,b}(x_b^c) + x_a^b \\ &= x_{a,c}(x_c) + x_{a,b}(x_b^c) + x_a^b \end{aligned}$$

and

$$x'_{a,c}(x_c) = x_{a,c}(x_c) + x_a^c.$$

Since $x'_{a,b} \circ x'_{b,c} = x'_{a,c}$, we get

$$x_{a,b}(x_b^c) + x_a^b = x_a^c.$$

Then, for $c \geq b$, we have successively

$$\begin{aligned} x'_{a,b}(X_b) &= x_{a,b}(X_b) + x_a^b \\ &= x_{a,b}(X_b) + x_{a,b}(x_b^c) + x_a^b \\ &= x_{a,b}(X_b) + x_a^c \\ &\subset B(\epsilon) + x_{a,c}(X_c) + x_a^c \\ &\subset B(\epsilon) + x'_{a,c}(X_c). \end{aligned}$$

It follows that

$$x'_{a,b}(X_b) \subset \left[x'_{a,c}(X_c)\right]_\epsilon \qquad \forall\, c \geq b.$$

Hence, the projective system

$$(X_a, x'_{a,b})_{a \in A}$$

satisfies the conditions of Proposition 4.3. Since for $b \geq a$, the diagram

$$\begin{tikzcd} X_b \arrow[r, "f_b"] \arrow[d, "x'_{a,b}"'] & M_b \arrow[d, "y_{a,b}"] \\ X_a \arrow[r, "f_a"'] & M_a \end{tikzcd}$$

commutes and since for any $a \in A$, $f_a$ is bijective, we may turn

$$(M_a, y_{a,b})_{a \in A}$$

into a projective system of complete non-empty metric spaces which satisfies the same conditions. Therefore,

$$\varprojlim_{a \in A} M_a \neq \emptyset.$$

Then, there is $m = (m_a)_{a \in A} \in \varprojlim\limits_{a \in A} M_a$ and we have

$$(\varprojlim_{a \in A} v_a)(m) = (v_a(m_a))_{a \in A} = (z_a)_{a \in A} = z.$$

□

**Theorem 4.6.** *Let $(X_a, x_{a,b})_{a \in A}$ be a filtering projective system of topological abelian groups. Assume that $A$ has a countable cofinal subset and that for any $a \in A$, $X_a$ is metrizable and complete. Then, $(X_a, x_{a,b})_{a \in A}$ is $\varprojlim\limits_{a \in A}$-acyclic if and only if for any $a \in A$ and any neighborhood of zero $V$ in $X_a$, there is $b \geq a$ such that*

$$x_{a,b}(X_b) \subset V + x_{a,c}(X_c) \qquad \forall\, c \geq b.$$

*Proof. The condition is sufficient.* It is clear that $\mathrm{cf}(A) \leq \omega_0$. Hence, by Theorem 2.10,

$$LH^k(\mathrm{R}\varprojlim_{a\in A} X_a) = 0 \qquad k \geq 2.$$

Moreover, there is a strict monomorphism

$$e : X \longrightarrow \Pi(\mathrm{O}(X)).$$

If $(Z, q)$ is the cokernel of $e$, the sequence

$$0 \longrightarrow X \xrightarrow{e} \Pi(\mathrm{O}(X)) \xrightarrow{q} Z \longrightarrow 0$$

is strictly exact and it gives rise to the long exact sequence

$$0 \longrightarrow \varprojlim_{a\in A} X_a \xrightarrow{\varprojlim_{a\in A} e_a} \varprojlim_{a\in A}(\Pi(\mathrm{O}(X)))_a \xrightarrow{\varprojlim_{a\in A} q_a} \varprojlim_{a\in A} Z_a \longrightarrow LH^1(\mathrm{R}\varprojlim_{a\in A} X_a) \longrightarrow 0 \qquad (*)$$

of $\mathcal{LH}(\mathcal{TA}b)$. Set

$$f = \varprojlim_{a\in A} q_a.$$

By Proposition 4.5, $f$ is surjective. Now, let us show that $f$ is strict.

For $b \geq a$, since $x_{a,b}$ is additive and continuous, it is uniformly continuous. Consider $a \in A$ and $\epsilon > 0$. By hypothesis, there is $b \geq a$ such that

$$x_{a,b}(X_b) \subset B(\epsilon) + x_{a,c}(X_c) \qquad \forall\, c \geq b.$$

It follows that

$$x_{a,b}(X_b) \subset [x_{a,c}(X_c)]_\epsilon \qquad \forall\, c \geq b.$$

Therefore, by Proposition 4.3, for any $a \in A$ and any $\epsilon > 0$, there is $b \geq a$ such that

$$x_{a,b}(X_b) \subset \left[ q_a(\varprojlim_{a\in A} X_a) \right]_\epsilon .$$

Consider $a \in A$ and $V$ a neighborhood of zero in $X_a$. There is $\epsilon > 0$ such that $V \supset B(\epsilon)$. By what precedes, there is $b \geq a$ such that

$$x_{a,b}(X_b) \subset \left[ q_a(\varprojlim_{a\in A} X_a) \right]_\epsilon .$$

Therefore,

$$x_{a,b}(X_b) \subset B(\epsilon) + q_a(\varprojlim_{a\in A} X_a) \subset V + q_a(\varprojlim_{a\in A} X_a)$$

and for $c \geq b$,

$$x_{a,c}(X_c) = x_{a,b}(x_{b,c}(X_c)) \subset x_{a,b}(X_b) \subset V + q_a(\varprojlim_{a\in A} X_a).$$

Then, by Theorem 3.4,

$$LH^1(\mathrm{R}\varprojlim_{a\in A} X_a) \in \mathcal{TA}b.$$

Let

$$J : \mathcal{TA}b \longrightarrow \mathcal{LH}(\mathcal{TA}b)$$

be the canonical functor. We know that the cokernel of $J(f)$ in $\mathcal{LH}(\mathcal{TA}b)$ is given by the complex

$$0 \longrightarrow \operatorname{coim} f \xrightarrow{f'} \varprojlim_{a\in A} Z_a \longrightarrow 0$$

where $\varprojlim_{a\in A} Z_a$ is in degree 0. Moreover, $f'$ is monomorphic and

$$\operatorname{coker} f \simeq \operatorname{coker} f'.$$

Hence, we get

$$\operatorname{coim} f \simeq \operatorname{coim} f' \qquad \text{and} \qquad \operatorname{im} f \simeq \operatorname{im} f'.$$

Since the sequence (*) is exact in $\mathcal{LH}(\mathcal{TA}b)$, we have

$$\operatorname{coker}(J(f)) \simeq LH^1(\mathrm{R}\varprojlim_{a\in A} X_a).$$

Therefore, $\operatorname{coker} J(f) \in \mathcal{TA}b$. Then, $f'$ is strict and it follows that so is $f$.

Finally, since $f$ is a strict epimorphism, we obtain

$$\operatorname{coker}(J(f)) \simeq LH^1(\mathrm{R}\varprojlim_{a\in A} X_a) \simeq 0$$

and

$$LH^k(\mathrm{R}\varprojlim_{a\in A} X_a) \simeq 0 \qquad \forall\, k \geq 1.$$

*The condition is necessary.* Since $(X_a, x_{a,b})_{a\in A}$ is $\varprojlim_{a\in A}$-acyclic,

$$LH^1(\mathrm{R}\varprojlim_{a\in A} X_a) \simeq 0 \in \mathcal{TA}b.$$

Then, by Theorem 3.4, for any $a \in A$ and any neighborhood of zero $V$ in $X_a$, there is $b \geq a$ such that

$$x_{a,c}(X_c) \subset V + q_a(\varprojlim_{a\in A} X_a) \qquad \forall\, c \geq b.$$

In particular,

$$x_{a,b}(X_b) \subset V + q_a(\varprojlim_{a \in A} X_a).$$

Since, for $c \geq b$, $x_{a,c} \circ q_c = q_a$, we have

$$x_{a,b}(X_b) \subset V + x_{a,c}(q_c(\varprojlim_{a \in A} X_a)) \subset V + x_{a,c}(X_c).$$

$\square$

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