Prépublications Mathématiques

de l'Université Paris 13

A Homological Study of Bornological Spaces

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00-21 December 2000

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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December 15, 2000

Abstract

In this paper, we show that the category $\mathcal{B}c$ of bornological vector spaces of convex type and the full subcategories $\widetilde{\mathcal{B}c}$ and $\widehat{\mathcal{B}c}$ formed by separated and complete objects form quasi-abelian categories. This allows us to study them from a homological point of view. In particular, we characterize acyclic inductive systems and prove that although the categories $\mathcal{B}c$ (resp. $\widetilde{\mathcal{B}c}$, $\mathcal{B}an$) and the categories $\mathcal{I}nd(\mathcal{S}ns)$ (resp. $\mathcal{I}nd(\mathcal{N}vs)$, $\mathcal{I}nd(\mathcal{B}an)$) formed by the ind-objects of the category of semi-normed vector spaces (resp. normed vector spaces, Banach spaces) are not equivalent, there is an equivalence between their derived categories given by a canonical triangulated functor which preserves the left t-structures. In particular these categories have the same left heart; a fact which means roughly that they have the same homological algebra. As a consequence, we get a link between the theory of sheaves of complete bornological spaces and that of sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$ used in [4].

0 Introduction

In this paper, we study from the point of view of homological algebra the category $\mathcal{B}c$ of bornological vector spaces of convex type and its full subcategories $\widetilde{\mathcal{B}c}$ and $\widehat{\mathcal{B}c}$ formed by separated and complete objects. Although these categories are not abelian, they are quasi-abelian and so we can take advantage of the tools developed in [5]. Our motivation for starting this work was to understand the link between these categories and the categories of $\mathcal{I}nd(\mathcal{S}ns)$ (resp. $\mathcal{I}nd(\mathcal{N}vs)$, $\mathcal{I}nd(\mathcal{B}an)$) formed by the ind-objects of the category $\mathcal{S}ns$ (resp. $\mathcal{N}vs$, $\mathcal{B}an$) of semi-normed vector spaces (resp. normed vector spaces, Banach spaces) used in [5] and in [4] to construct convenient categories of topological sheaves. Our main is being that although the categories $\mathcal{B}c$ (resp. $\widetilde{\mathcal{B}c}$, $\mathcal{B}an$) and $\mathcal{I}nd(\mathcal{S}ns)$ (resp. $\mathcal{I}nd(\mathcal{N}vs)$, $\mathcal{I}nd(\mathcal{B}an)$) are

1991 AMS Mathematics Subject Classification. 46M20, 46M40, 18G50.

Key words and phrases. Homological methods for functional analysis, derived inductive limits, ind-objects, non-abelian homological algebra, quasi-abelian categories.

not equivalent, there is an equivalence between their derived categories given by a canonical triangulated functor which preserves the left t-structures. In particular these categories have the same left heart; a fact which means roughly that they have the same homological algebra. Note that the left heart $\mathcal{LH}(\mathcal{B}c)$ associated to $\mathcal{B}c$ coincide with the category of "quotient bornological spaces" considered by L. Waelbroek some years ago. Note also that among our results is the somewhat astonishing fact that the left heart $\mathcal{LH}(\widehat{\mathcal{B}c})$ associated to $\widehat{\mathcal{B}c}$ is equivalent to $\mathcal{LH}(\mathcal{B}c)$.

Let us now describe with some details the contents of this paper.

We begin Section 1 by recalling a few basic facts about bornological vector spaces of convex type and by establishing that they form a quasi-abelian category. We also characterize the strict morphisms of this category and prove that direct sums and direct products are kernel and cokernel preserving; a fact which will allow us to apply the results of [2] to study derived inductive limits in $\mathcal{B}c$.

Section 2 is devoted to the properties of the filtering inductive systems of $\mathcal{B}c$. Our first result is an explicit characterization of acyclic ones by means of condition SC' (a kind of dual Mittag-Leffler condition). We also establish that semi-normed spaces are small as objects of $\mathcal{B}c$ and that $\mathcal{B}c$ has enough projective objects.

In Section 3, we consider the relation between $\mathcal{B}c$ and $\mathcal{I}nd(\mathcal{S}ns)$. We show first that $\mathcal{B}c$ may be identified with the full subcategory of $\mathcal{I}nd(\mathcal{S}ns)$ formed by the essentially monomorphic objects. Next, we establish that $\mathcal{B}c$ and $\mathcal{I}nd(\mathcal{S}ns)$ generate equivalent derived categories and equivalent left hearts.

The study of the category $\widehat{\mathcal{B}c}$ of separated objects of $\mathcal{B}c$ is performed in Section 4. We show first that this category is quasi-abelian, that it has enough projective objects, that direct sums are cokernel preserving and that direct sums are strictly exact. Next, we consider the inclusion of $\widehat{\mathcal{B}c}$ into $\mathcal{B}c$ and its left adjoint and show that they induce inverse equivalences at the level of derived categories. Finally, we identify $\widehat{\mathcal{B}c}$ with the full subcategory of $\mathcal{I}nd(\mathcal{N}vs)$ formed by the essentially monomorphic objects. We conclude by showing that $\widehat{\mathcal{B}c}$ and $\mathcal{I}nd(\mathcal{N}vs)$ generate equivalent derived categories and equivalent left hearts.

Section 5 is concerned with completeness. We follow roughly the same plan as in Section 4 and get the same type of result but of the proofs are slightly more involved. As a bonus, we gain a rather good understanding of the derived functor of the completion functor $\widehat{\mathrm{Cpl}} : \mathcal{B}c \to \widehat{\mathcal{B}c}$.

In the Appendix, we have gathered a few facts about the essentially monomorphic ind-objects of a quasi-abelian category which are well-known in the abelian case but which lacked a proof in the quasi-abelian one.

To conclude, let us point out that a study of the homological algebra of the category of locally convex topological vector spaces using similar techniques was performed in [3].

1 The category $\mathcal{B}c$

Definition 1.1. We denote by $\mathcal{B}c$ the category of bornological $\mathbb{C}$-vector spaces of convex type in the sense of Houzel [1].

An *object E of $\mathcal{B}c$* is thus a $\mathbb{C}$-vector space endowed with a *bornology of convex type* i.e. a family $\mathcal{B}_E$ of subsets of E such that

(a) if $B \in \mathcal{B}$ and $B' \subset B$ then $B' \in \mathcal{B}$;

(b) if B_1, $B_2 \in \mathcal{B}$ then $B_1 \cup B_2 \in \mathcal{B}$;

(c) if $B \in \mathcal{B}$ and $\lambda > 0$ then $\lambda B \in \mathcal{B}$;

(d) if $B \in \mathcal{B}$ then the absolutely convex hull $\langle B \rangle$ of B belongs to $\mathcal{B}$;

(e) for any $x \in E$, $\{x\} \in \mathcal{B}$.

For short, we call the elements of $\mathcal{B}_E$ the *bounded subsets* of E.

A *morphism of $\mathcal{B}c$* is a $\mathbb{C}$-linear map

$$u : E \to F$$

such that $u(B)$ is bounded in F for any B bounded in E.

It is well-known that the category $\mathcal{B}c$ has direct sums and direct products. Their structure is recalled hereafter.

Proposition 1.2. *Let $(E_i)_{i \in I}$ be a small family of objects of $\mathcal{B}c$. Then,*

(a) the $\mathbb{C}$-vector space

$$\bigoplus_{i \in I} E_i$$

endowed with the bornology

$$\{B : B \subset \bigoplus_{i \in I} B_i, B_i \text{ bounded in } E_i \text{ for any } i \in I, \{i \in I : B_i \neq \{0\}\} \text{ finite}\}$$

together with the canonical morphisms

$$s_i : E_i \to \bigoplus_{i \in I} E_i$$

is a direct sum of $(E_i)_{i \in I}$ in $\mathcal{B}c$,

(b) *the* $\mathbb{C}$*-vector space*

$$\prod_{i\in I} E_i$$

endowed with the bornology

$$\{B : B \subset \prod_{i\in I} B_i, B_i \text{ bounded in } E_i \text{ for any } i \in I\}$$

together with the canonical morphisms

$$p_i : \prod_{i\in I} E_i \to E_i$$

is a direct product of $(E_i)_{i\in I}$.

Remark 1.3. The notation

$$\bigoplus_{i\in I} B_i$$

used in the preceding proposition means as usual

$$\{\sum_{i\in I} s_i(b_i) : b_i \in B_i \text{ for any } i \in I\};$$

this has a meaning since $\{i \in I : B_i \neq \{0\}\}$ is finite. As for the notation

$$\prod_{i\in I} B_i,$$

it means of course

$$\{x \in \prod_{i\in I} E_i : p_i(x) \in B_i \text{ for any } i \in I\}.$$

Definition 1.4. Let E be an object of $\mathcal{B}c$, let F be a $\mathbb{C}$-vector subspace of E. The set

$$\{B \cap F : B \text{ bounded in } E\}$$

is clearly a bornology on F. We call it the *induced bornology.*

Similarly, if $q : E \to E/F$ denotes the canonical morphism, the set

$$\{q(B) : B \text{ bounded in } E\}$$

forms a bornology on E/F called the *quotient bornology.*

With this definition at hand, one checks easily that

Proposition 1.5. *The category $\mathcal{B}c$ is additive. Moreover, if $u : E \to F$ is a morphism of $\mathcal{B}c$, then*

(a) $\operatorname{Ker} u$ *is the subspace $u^{-1}(0)$ of E endowed with the induced bornology;*

(b) $\operatorname{Coker} u$ *is the quotient space $F/u(E)$ endowed with the quotient bornology;*

(c) $\operatorname{Im} u$ *is the subspace $u(E)$ of F endowed with the induced bornology;*

(d) $\operatorname{Coim} u$ *is the quotient space $E/u^{-1}(0)$ endowed with the quotient bornology.*

As a consequence we have the following characterization of the strict morphisms of $\mathcal{B}c$.

Proposition 1.6. *A morphism $u : E \to F$ of $\mathcal{B}c$ is strict if and only if for any bounded subset B of F, there is a bounded subset B' of E such that*

$$B \cap u(E) = u(B').$$

Proof. By definition, $u : E \to F$ is strict if and only if the canonical morphism $\tilde{u} : \operatorname{Coim} u \to \operatorname{Im} u$ is an isomorphism. Thanks to the preceding proposition, this canonical morphism is the morphism

$$\tilde{u} : E/u^{-1}(0) \to u(E)$$

defined by

$$\tilde{u}([e]_{u^{-1}(0)}) = u(e) \qquad \forall\, e \in E.$$

It is thus a bijective linear map. Therefore, it will be an isomorphism in $\mathcal{B}c$ if and only if it exchanges the quotient bornology on $E/u^{-1}(0)$ with the induced bornology on $u(E)$. Going back to the definition of these bornologies, we get the conclusion. □

Corollary 1.7. *Let $u : E \to F$ be a morphism of $\mathcal{B}c$. Then,*

(a) u is a strict monomorphism if and only if u is an injective map such that $u^{-1}(B)$ is a bounded subset of E for any bounded subset B of F;

(b) u is a strict epimorphism if and only if u is a surjective map such that for any bounded subset B of F there is a bounded subset B' of E with $B = u(B')$.

Proposition 1.8. *The category $\mathcal{B}c$ is quasi-abelian.*

Proof. We know that $\mathcal{B}c$ is additive and has kernels and cokernels.

(i) Consider a cartesian square

$$\begin{array}{ccc} M_0 & \xrightarrow{u} & N_0 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ M_1 & \xrightarrow[v]{} & N_1 \end{array}$$

where u is a strict epimorphism and let us show that v is a strict epimorphism. Recall that if we set

$$\alpha = \begin{pmatrix} u & -g \end{pmatrix} : M_0 \oplus N_1 \to N_0,$$

then we may assume that

$$M_1 = \operatorname{Ker} \alpha = \{(m_0, n_1) : u(m_0) = g(n_1)\}$$

and that

$$f = p_{M_0} \circ i_\alpha \quad \text{and} \quad v = p_{N_1} \circ i_\alpha$$

where $i_\alpha : \operatorname{Ker} \alpha \to M_0 \oplus N_1$ is the canonical monomorphism and p_{M_0}, p_{N_1} are the canonical morphisms associated to the direct product $M_0 \oplus N_1$.

Of course, the morphism v is surjective. Let us prove that it is strict. Consider a bounded subset B_1 of N_1. Since $g(B_1)$ is a bounded subset of N_0 and since u is strict, there is a bounded subset B_0 of M_0 such that

$$g(B_1) = u(B_0).$$

Since $p_{M_0}^{-1}(B_0) \cap p_{N_1}^{-1}(B_1)$ is a bounded subset of $M_0 \oplus N_1$,

$$\alpha^{-1}(0) \cap p_{M_0}^{-1}(B_0) \cap p_{N_1}^{-1}(B_1)$$

is a bounded subset of M_1 and a direct computation shows that

$$v\left(\alpha^{-1}(0) \cap p_{M_0}^{-1}(B_0) \cap p_{N_1}^{-1}(B_1)\right) \supset B_1.$$

The conclusion follows.

(ii) Consider a cocartesian square

$$\begin{array}{ccc} M_1 & \xrightarrow{v} & N_1 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ M_0 & \xrightarrow[u]{} & N_0 \end{array}$$

where u is a strict monomorphism. Let us show that v is a strict monomorphism. Recall that if we set

$$\alpha = \begin{pmatrix} f \\ -u \end{pmatrix} : M_0 \to M_1 \oplus N_0,$$

then we may assume that

$$N_1 = \operatorname{Coker} \alpha = (M_1 \oplus N_0)/\alpha(M_0),$$

and that

$$v = q_\alpha \circ s_{M_1} \qquad \text{and} \qquad g = q_\alpha \circ s_{N_0}$$

where $q_\alpha : M_1 \oplus N_0 \to (M_1 \oplus N_0)/\alpha(M_0)$ is the canonical epimorphism and s_{M_1}, s_{N_0} are the canonical morphisms associated to the direct sum $M_1 \oplus N_0$.

Clearly, the morphism v is injective. Let us prove that it is strict. Let B_1 be a bounded subset of N_1 and let B be a bounded subset of $M_1 \oplus N_0$ such that $B_1 = q_\alpha(B)$. It is sufficient to show that $v^{-1}(q_\alpha(B))$ is bounded in M_1. Since B is a bounded subset of $M_1 \oplus N_0$, there are absolutely convex bounded subsets B_1', B_0 of M_1 and N_0 such that

$$B \subset B_1' \oplus B_0.$$

Since u is a strict monomorphism, $B_0' = u^{-1}(B_0)$ is an absolutely convex bounded subset of M_0. A simple computation shows that

$$v^{-1}(q_\alpha(B)) \subset B_1' + f(B_0') \subset 2 \langle B_1' \cup f(B_0') \rangle$$

and the conclusion follows. □

Proposition 1.9. *The category $\mathcal{B}c$ is complete and cocomplete. Moreover, direct sums and direct products are kernel and cokernel preserving.*

Proof. Since $\mathcal{B}c$ has kernels and direct products, (resp. cokernels and direct sums), $\mathcal{B}c$ has projective (resp. inductive) limits. Hence, $\mathcal{B}c$ is complete (resp. cocomplete).

We know that direct sums are cokernel preserving. Let us show that they are kernel preserving. Consider a family $(u_i : E_i \to F_i)_{i \in I}$ of morphisms of $\mathcal{B}c$. For any $i \in I$, denote K_i the kernel of u_i and $k_i : K_i \to E_i$ the canonical morphism. It is clear that

$$0 \to \bigoplus_{i \in I} K_i \xrightarrow{\bigoplus_{i \in I} k_i} \bigoplus_{i \in I} E_i \xrightarrow{\bigoplus_{i \in I} u_i} \bigoplus_{i \in I} F_i$$

is an exact sequence of vector spaces. Let us show that it is strictly exact. Let B be a bounded subset of $\bigoplus_{i \in I} E_i$. By Corollary 1.7, it is sufficient to show that

$$(\bigoplus_{i \in I} k_i)^{-1}(B)$$

is bounded in $\bigoplus_{i \in I} K_i$. We know that

$$B \subset \bigoplus_{i \in I} B_i$$

where B_i is a bounded subset of E_i and the set $A = \{i : B_i \neq \{0\}\}$ is finite. Moreover, $k_i^{-1}(B_i)$ is a bounded subset of K_i for any $i \in I$ and $k_i^{-1}(B_i) = \{0\}$ for $i \in I \setminus A$ since k_i is injective. It follows that

$$\bigoplus_{i \in I} k_i^{-1}(B_i)$$

is a bounded subset of $\bigoplus_{i \in I} K_i$. Since

$$(\bigoplus_{i \in I} k_i)^{-1}(B) \subset \bigoplus_{i \in I} k_i^{-1}(B_i)$$

the conclusion follows.

A similar reasoning gives the result for direct products. □

2 Filtering inductive limits in $\mathcal{B}c$

Let I be a small filtering preordered set. Hereafter, we will often view I as a small category. This allows us to identify the category of inductive systems of objects of $\mathcal{B}c$ indexed by I with the category $\mathcal{B}c^I$ of functors from I to $\mathcal{B}c$. If E is an object of this category, we will denote by (E_i, e_{ij}) the corresponding inductive system and we will denote r_i the canonical morphism

$$E_i \to \varinjlim_{i \in I} E_i.$$

The following result is easily checked.

Proposition 2.1. *Let I be a small filtering preordered set and let E be an object of $\mathcal{B}c^I$. Then, the bornology on*

$$\varinjlim_{i \in I} E_i$$

is formed by the sets B such that

$$B \subset r_i(B_i)$$

for some $i \in I$ and some bounded subset B_i of E_i.

Definition 2.2. Let$I$ be a small filtering preordered set. We say that an inductive system $E \in \mathcal{B}c^I$ satisfies *condition SC'* if for any $i \in I$ and any bounded subset B of E_i, there is $j \geq i$ such that

$$B \cap \operatorname{Ker} r_i \subset \operatorname{Ker} e_{ji}$$

or, equivalently, if the sequence

$$(B \cap \operatorname{Ker} e_{ji})_{j \geq i}$$

of subsets of E_i is stationary.

Hereafter, we will use freely the theory of the derivation of inductive limits in quasi-abelian categories developed in [2].

Lemma 2.3. *Let I be a small filtering preordered set and let*

$$0 \to E' \xrightarrow{u'} E \xrightarrow{u''} E'' \to 0$$

be a strictly exact sequence of $\mathcal{B}c^I$. Assume that E'' satisfies condition SC'. Then,

$$0 \to \varinjlim_{i\in I} E'_i \xrightarrow{\varinjlim_{i\in I} u'_i} \varinjlim_{i\in I} E_i \xrightarrow{\varinjlim_{i\in I} u''_i} \varinjlim_{i\in I} E''_i \to 0$$

is a strictly exact sequence of $\mathcal{B}c$.

Proof. We already know that the considered sequence is an exact sequence of vector spaces. Since inductive limits are cokernel preserving, it remains to prove that $\varinjlim_{i\in I} u'_i$ is strict. Let B be an arbitrary bounded subset of $\varinjlim_{i\in I} E_i$. By Corollary 1.7, it is sufficient to show that $(\varinjlim_{i\in I} u'_i)^{-1}(B)$ is a bounded subset of $\varinjlim_{i\in I} E'_i$. By Proposition 2.1, we know that there is $i \in I$ and a bounded subset B_i of E_i such that $B \subset r_i(B_i)$. Set

$$C_i = B_i \cap r_i^{-1}(\operatorname{Ker}(\varinjlim_{i\in I} u''_i)).$$

Since $u''_i(C_i)$ is a bounded subset of E''_i such that

$$r''_i(u''_i(C_i)) = 0,$$

it follows from our assumption that there is $j \geq i$ such that

$$e''_{ji}(u''_i(C_i)) = 0.$$

Therefore,

$$e_{ji}(C_i) \subset \operatorname{Ker} u''_j$$

and since u'_j is a strict monomorphism of $\mathcal{B}c$, there is a bounded subset B'_j of E'_j such that

$$e_{ji}(C_i) \subset u'_j(B'_j).$$

Hence for any $k \geq j$,

$$e_{ki}(C_i) \subset u'_k(e'_{kj}(B'_j))$$

and

$$r'_k({u'_k}^{-1}(e_{ki}(C_i))) \subset r'_j(B'_j).$$

Since one checks easily that

$$(\varinjlim_{i\in I} u'_i)^{-1}(B) \subset \bigcup_{k\geq j} r'_k({u'_k}^{-1}(e_{ki}(C_i))),$$

it follows that

$$(\varinjlim_{i\in I} u'_i)^{-1}(B) \subset r'_j(B_j).$$

Therefore

$$(\varinjlim_{i\in I} u'_i)^{-1}(B)$$

is a bounded subset of $\varinjlim_{i\in I} E'_i$ and the proof is complete. □

Lemma 2.4. *Let I be a small filtering preordered set. Then,*

(a) any object of $\mathcal{B}c^I$ of coproduct type satisfies condition SC';

(b) if $E \to P$ is a monomorphism in $\mathcal{B}c^I$ and if P satisfies condition SC', then E also satisfies condition SC'.

(c) In particular, the full subcategory of $\mathcal{B}c^I$ formed by the objects satisfying condition SC' is a $\varinjlim_{i\in I}$-projective subcategory.

Proof.

(a) This follows from the fact that the transitions of an inductive system of coproduct type are monomorphic.

(b) This is obvious.

(c) This follows from the preceding lemma combined with (a) and (b) and the fact that any object of $\mathcal{B}c^I$ is a strict quotient of an object of coproduct type.

□

Lemma 2.5. *Let I be a small filtering preordered set and let E be an object of $\mathcal{B}c^I$. If*

$$RH_1(\operatorname{L}\varinjlim_{i\in I} E_i) \in \mathcal{B}c,$$

then E satisfies condition SC'.

Proof. Since direct sums are exact in $\mathcal{B}c$, $\operatorname{L}\varinjlim_{i\in I} E_i$ is represented by the negative Roos complex $R.(I,E)$. Our assumption means that the differential

$$\begin{array}{ccc} R_1(I,E) & \xrightarrow{d} & R_0(I,E) \\ \| & & \| \\ \bigoplus_{i\le j} E_i & \longrightarrow & \bigoplus_i E_i \end{array}$$

is strict. Recall that

$$d \circ s_{ij} = s_j \circ e_{ji} - s_i \qquad (i \le j)$$

where $s_{ij} : E_i \to \bigoplus_{i\le j} E_i$, $s_i : E_i \to \bigoplus_i E_i$ denote the canonical inclusions. Recall also that the augmentation

$$\epsilon : R_0(I,E) \to \varinjlim_{i\in I} E_i$$

is defined by

$$\epsilon \circ s_i = r_i \qquad \forall i \in I.$$

Let $i_0 \in I$, let B be a bounded subset of E_{i_0} and consider the bounded subset

$$s_{i_0}(B \cap r_{i_0}^{-1}(0))$$

of $\bigoplus_{i\in I} E_i$. By construction $\epsilon(s_{i_0}(B \cap r_{i_0}^{-1}(0))) = 0$, hence $s_{i_0}(B \cap r_{i_0}^{-1}(0)) \subset \operatorname{Im} d$. Since d is assumed to be strict, there is a bounded subset B' of $\bigoplus_{i\le j} E_i$ such that

$$s_{i_0}(B \cap r_{i_0}^{-1}(0)) \subset d(B').$$

Using the structure of bornology of $\bigoplus_{i\le j} E_i$, we see that

$$B' \subset \bigoplus_{i\le j} B'_{ij}$$

where B'_{ij} is a bounded subset of E_i for any $i \le j$, the set

$$S = \{(i,j) : i \le j, \quad B'_{ij} \neq \{0\}\}$$

being finite. Since I is filtering, there is an element k of I such that $k \geq i_0$ and $k \geq j$ for any $(i,j) \in S$. Denote

$$\pi : \bigoplus_{i} E_i \to \bigoplus_{i \leq k} E_i$$

the canonical projection and

$$\tau : \bigoplus_{i \leq k} E_i \to E_k$$

the morphism defined by setting

$$\tau \circ \sigma_i = e_{ki} \qquad (i \leq k)$$

where $\sigma_i : S_i \to \bigoplus_{i \leq k} S_i$ is the canonical inclusion. It follows from the definition of d and k that

$$\tau \circ \pi \circ d(B') = 0.$$

Hence

$$\tau \circ \pi(s_{i_0}(B \cap r_{i_0}^{-1}(0))) = 0.$$

Since

$$\tau \circ \pi \circ s_{i_0} = \tau \circ \sigma_{i_0} = e_{ki_0},$$

the proof is complete. □

Proposition 2.6. *Let I be a small filtering preordered set and let E be an object of $\mathcal{B}c^I$. Then, the following conditions are equivalent:*

(a) E is $\varinjlim_{i \in I}$-acyclic (i.e. the canonical morphism

$$\mathrm{L}\varinjlim_{i \in I} E_i \to \varinjlim_{i \in I} E_i$$

is an isomorphism in $D^-(\mathcal{B}c)$);

(b) E satisfies condition SC'.

Proof.
(a) ⇒ (b). This is a direct consequence of Lemma 2.5.
(b) ⇒ (a). This follows from part (c) of Lemma 2.4. □

Corollary 2.7. *Let I be a small filtering preordered set and let E be an object of $\mathcal{B}c^I$. Then*

$$RH_k(\mathrm{L}\varinjlim_{i \in I} E_i) = 0 \qquad \text{for } k \geq 2.$$

Proof. Using Lemma 2.4, we see directly that E has a resolution of the form

$$0 \to P_1 \to P_0 \to E \to 0$$

where P_0, P_1 satisfy condition SC'. The conclusion follows easily. □

Definition 2.8. For any object E of $\mathcal{B}c$, $\tilde{\mathcal{B}}_E$ the set of absolutely convex bounded subsets of E. Of course, the inclusion $\tilde{\mathcal{B}}_E \subset \mathcal{B}_E$ is cofinal. If $B \in \tilde{\mathcal{B}}_E$, we denote E_B the semi-normed space obtained by endowing the linear hull of B with the gauge semi-norm p_B defined by setting

$$p_B(x) = \inf\{\lambda \geq 0 : x \in \lambda B\}$$

for any x in E_B. When considered as a bornological space, E_B will always be endowed with its canonical bornology i.e.

$$\{B' : B' \subset \lambda B \text{ for some } \lambda \geq 0\}.$$

Proposition 2.9. *For any object E of $\mathcal{B}c$, the canonical morphism*

$$u : \varinjlim_{B \in \tilde{\mathcal{B}}_E} E_B \to E$$

is an isomorphism.

Proof. Consider the map

$$v : E \to \varinjlim_{B \in \tilde{\mathcal{B}}_E} E_B$$

defined by setting

$$v(e) = r_{\langle\{e\}\rangle}(e) \qquad \forall e \in E.$$

A direct computation shows that v is linear. Moreover, the image by v of a bounded subset B of E is bounded since

$$v(B) \subset r_{\langle B\rangle}(\langle B\rangle).$$

It follows that v is a morphism of $\mathcal{B}c$. By construction $u \circ v = \mathrm{id}$ and one checks easily that $v \circ u = \mathrm{id}$; hence the conclusion. □

Proposition 2.10. *Let I be a small filtering preordered set, let E be an object of $\mathcal{B}c^I$ and let F be an object of $\mathcal{S}ns$. Assume E satisfies condition SC'. Then, the canonical morphism*

$$\varinjlim_{i \in I} \operatorname{Hom}_{\mathcal{B}c}(F, E_i) \to \operatorname{Hom}_{\mathcal{B}c}(F, \varinjlim_{i \in I} E_i)$$

is an isomorphism.

Proof. Let φ be the morphism considered in the statement. Denote

$$r_i : E_i \to \varinjlim_{i\in I} E_i$$

and

$$r'_i : \operatorname{Hom}_{\mathcal{B}c}(F, E_i) \to \varinjlim_{i\in I} \operatorname{Hom}_{\mathcal{B}c}(F, E_i)$$

the canonical morphisms. Recall that φ is characterized by the fact that

$$\varphi(r'_i(h_i)) = r_i \circ h_i$$

for any $h_i \in \operatorname{Hom}_{\mathcal{B}c}(F, E_i)$.

Let us show that φ is injective. Fix $h_i : F \to E_i$ and assume $\varphi(r'_i(h_i)) = 0$. It follows from the formula recalled above that $r_i \circ h_i = 0$. Denote b_F the unit ball of F. We have $h_i(b_F) \subset r_i^{-1}(0)$ and, since E satisfies condition SC', there is $j \geq i$ such that $h_i(b_F) \subset e_{ji}^{-1}(0)$. For such a j, we have by linearity $e_{ji} \circ h_i = 0$. So $r'_i(h_i) = 0$ as expected.

Let us now prove the surjectivity of φ. Fix

$$h : F \to \varinjlim_{i\in I} E_i.$$

Since $h(b_F)$ is a bounded subset of the inductive limit, there is $i \in I$ and an absolutely convex bounded subset B_i of E_i such that

$$h(b_F) \subset r_i(B_i).$$

Using our assumptions, we find $j \geq i$ such that

$$B_i \cap r_i^{-1}(0) \subset B_i \cap e_{ji}^{-1}(0).$$

Set $B_j = e_{ji}(B_i)$ and $L_j = (E_j)_{B_j}$. By construction,

$$r_j : L_j \to \varinjlim_{i\in I} E_i$$

is injective and

$$h(F) \subset r_j(L_j).$$

It follows that there is a unique linear map $h_j : F \to L_j$ such that $r_j \circ h_j = h$. Since $h_j(b_F) \subset B_j$, this map is in fact a morphism of $\mathcal{B}c$. Composing h_j with the inclusion of L_j into E_j, we get a morphism $h'_j : F \to E_j$ such that $\varphi(r'_j(h'_j)) = h$; hence the conclusion. □

Corollary 2.11. *Any semi-normed space E is a small object of $\mathcal{B}c$. More explicitly, for any family $(E_i)_{i\in I}$ of $\mathcal{B}c$ the canonical morphism*

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{B}c}(E, E_i) \to \mathrm{Hom}_{\mathcal{B}c}(E, \bigoplus_{i\in I} E_i)$$

is an isomorphism.

Proof. We have only to note that

$$\bigoplus_{i\in I} E_i = \varinjlim_{J\in\mathcal{P}_f(I)} \bigoplus_{j\in J} E_j$$

where $\mathcal{P}_f(I)$ is the ordered set formed by the finite subsets of I. □

Lemma 2.12. *Any projective object of $\mathcal{S}ns$ is projective in $\mathcal{B}c$.*

Proof. Let P be a projective object of $\mathcal{S}ns$ and let $f : E \to F$ be a strict epimorphism of $\mathcal{B}c$. For any $B \in \tilde{\mathcal{B}}_E$, $f : E_B \to F_{f(B)}$ is clearly a strict epimorphism of $\mathcal{B}c$ and hence of $\mathcal{S}ns$. Therefore, the sequences

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{S}ns}(P, E_B) & \longrightarrow & \mathrm{Hom}_{\mathcal{S}ns}(P, F_{f(B)}) & \longrightarrow & 0 \\ \downarrow \wr & & \downarrow \wr & & \\ \mathrm{Hom}_{\mathcal{B}c}(P, E_B) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}c}(P, F_{f(B)}) & \longrightarrow & 0 \end{array}$$

are exact. Inductive limits being exact in the category of abelian groups, the sequence

$$\varinjlim_{B\in\tilde{\mathcal{B}}_E} \mathrm{Hom}_{\mathcal{B}c}(P, E_B) \to \varinjlim_{B\in\tilde{\mathcal{B}}_E} \mathrm{Hom}_{\mathcal{B}c}(P, F_{f(B)}) \to 0$$

is also exact. By Proposition 2.10 combined with Proposition 2.9, we have

$$\varinjlim_{B\in\tilde{\mathcal{B}}_E} \mathrm{Hom}_{\mathcal{B}c}(P, E_B) \simeq \mathrm{Hom}_{\mathcal{B}c}(P, E).$$

Since $f : E \to F$ is a strict epimorphism, by Corollary 1.7, the inclusion

$$\{f(B) : B \in \tilde{\mathcal{B}}_E\} \subset \tilde{\mathcal{B}}_F$$

is cofinal. It follows that

$$\varinjlim_{B\in\tilde{\mathcal{B}}_E} \mathrm{Hom}_{\mathcal{B}c}(P, F_{f(B)}) \simeq \varinjlim_{B\in\tilde{\mathcal{B}}_F} \mathrm{Hom}_{\mathcal{B}c}(P, F_B) \simeq \mathrm{Hom}_{\mathcal{B}c}(P, F).$$

Therefore, the sequence

$$\mathrm{Hom}_{\mathcal{B}c}(P, E) \to \mathrm{Hom}_{\mathcal{B}c}(P, F) \to 0$$

is exact and P is projective in $\mathcal{B}c$. □

Proposition 2.13. *The category $\mathcal{B}c$ has enough projective objects.*

Proof. Let E be an object of $\mathcal{B}c$. Consider the canonical morphism

$$\bigoplus_{B\in\tilde{\mathcal{B}}_E} E_B \to E$$

induced by the inclusions

$$E_B \to E.$$

This is clearly a strict epimorphism. Since $\mathcal{S}ns$ has enough projective objects ([5, Proposition 3.2.11]), for any $B \in \tilde{\mathcal{B}}_E$, there is a strict epimorphism of the form

$$P_B \to E_B$$

where P_B is a projective object of $\mathcal{S}ns$. By Lemma 2.12, P_B is still projective in $\mathcal{B}c$. Direct sums being cokernel preserving in $\mathcal{B}c$,

$$\bigoplus_{B\in\tilde{\mathcal{B}}_E} P_B \to \bigoplus_{B\in\tilde{\mathcal{B}}_E} E_B$$

is a strict epimorphism. By composition, we get a strict epimorphism

$$\bigoplus_{B\in\tilde{\mathcal{B}}_E} P_B \to E$$

and the conclusion follows from the fact that a direct sum of projective objects is a projective object. □

3 Relations between $\mathcal{B}c$ and $\mathcal{I}nd(\mathcal{S}ns)$

Definition 3.1. Let

$$\mathrm{S} : \mathcal{B}c \to \mathcal{I}nd(\mathcal{S}ns)$$

be the functor which associates to any object E of $\mathcal{B}c$ the object

$$\text{“}\varinjlim_{B\in\tilde{\mathcal{B}}_E}\text{”}\, E_B$$

of $\mathcal{I}nd(\mathcal{S}ns)$ and let

$$\mathrm{L} : \mathcal{I}nd(\mathcal{S}ns) \to \mathcal{B}c$$

be the functor defined by setting

$$\mathrm{L}(\text{“}\varinjlim_{i\in I}\text{”}\, E_i) = \varinjlim_{i\in I} E_i$$

for any functor $E : I \to \mathcal{S}ns$.

Remark 3.2. If E is an object of $\mathcal{S}ns$, then

$$\mathrm{S}(E) \simeq \text{“}E\text{”}.$$

As a matter of fact, if b_E is the unit ball of E, we know that

$$\{rb_E : r > 0\}.$$

is a cofinal subset of $\tilde{\mathcal{B}}_E$. Since for any $r > 0$, we have $E_{rb_E} \simeq E_{b_E} \simeq E$, the conclusion follows.

Proposition 3.3.

(a) For any object X of $\mathcal{S}ns$ and any object E of $\mathcal{B}c$, we have

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}X\text{”}, \mathrm{S}(E)) \simeq \mathrm{Hom}_{\mathcal{B}c}(X, E).$$

(b) If $(E_i)_{i\in I}$ is a small family of $\mathcal{B}c$, then

$$\mathrm{S}(\bigoplus_{i\in I} E_i) \simeq \bigoplus_{i\in I} \mathrm{S}(E_i).$$

Proof. (a) We have successively

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}X\text{”}, \mathrm{S}(E)) &\simeq \mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}X\text{”}, \text{“}\varinjlim_{B\in\tilde{\mathcal{B}}_E}\text{”}\, E_B) \\
&\simeq \varinjlim_{B\in\tilde{\mathcal{B}}_E} \mathrm{Hom}_{\mathcal{S}ns}(X, E_B) \\
&\simeq \mathrm{Hom}_{\mathcal{B}c}(X, \varinjlim_{B\in\tilde{\mathcal{B}}_E} E_B)
\end{aligned}$$

where the last isomorphism comes from Proposition 2.10. Combining this with Proposition 2.9 gives us the announced result.

(b) For any object X of $\mathcal{S}ns$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}X\text{”}, \mathrm{S}(\bigoplus_{i\in I} E_i)) &\simeq \mathrm{Hom}_{\mathcal{B}c}(X, \bigoplus_{i\in I} E_i) \\
&\simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{B}c}(X, E_i) \qquad (*) \\
&\simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}X\text{”}, \mathrm{S}(E_i)) \\
&\simeq \mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}X\text{”}, \bigoplus_{i\in I} \mathrm{S}(E_i))
\end{aligned}$$

where the isomorphism (*) follows from Corollary 2.11 and the last isomorphism comes from [2, Proposition 7.1.9]. ☐

Proposition 3.4. *Let E be an object of $\mathcal{B}c$ and let F be an object of $\mathcal{I}nd(\mathcal{S}ns)$. Then,*

(a) there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{B}c}(\mathrm{L}(F), E) = \mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(F, \mathrm{S}(E));$$

(b) the canonical morphism

$$\mathrm{L} \circ \mathrm{S}(E) \to E$$

is an isomorphism;

(c) the following conditions are equivalent:

(i) the canonical morphism $F \to \mathrm{S} \circ \mathrm{L}(F)$ is an isomorphism;

(ii) F is in the essential image of S;

(iii) F is essentially monomorphic.

Proof. Assume $F = \text{“}\varinjlim_{i \in I}\text{”}\, F_i$

(a) We have successively

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}c}(\mathrm{L}(F), E) &\simeq \mathrm{Hom}_{\mathcal{B}c}(\varinjlim_{i \in I} F_i, E) \\
&\simeq \varprojlim_{i \in I} \mathrm{Hom}_{\mathcal{B}c}(F_i, E) \\
&\simeq \varprojlim_{i \in I} \varinjlim_{B \in \tilde{\mathcal{B}}_E} \mathrm{Hom}_{\mathcal{B}c}(F_i, E_B) \qquad (*) \\
&\simeq \mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(\text{“}\varinjlim_{i \in I}\text{”}\, F_i, \text{“}\varinjlim_{B \in \tilde{\mathcal{B}}_E}\text{”}\, E_B) \\
&\simeq \mathrm{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(F, \mathrm{S}(E))
\end{aligned}$$

where the isomorphism (*) follows from Proposition 2.10 and Proposition 2.9

(b) This is another way to state Proposition 2.9.

(c) It is sufficient to prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). Assume $F \simeq \mathrm{S}(E)$. Then,

$$F \simeq \text{“}\varinjlim_{B \in \tilde{\mathcal{B}}_E}\text{”}\, E_B.$$

Since $i_{B'B} : E_B \to E_{B'}$ is the canonical inclusion, $(E_B, i_{B'B})_{B \in \tilde{\mathcal{B}}_E}$ is an inductive system with injective transitions. Hence, F is essentially monomorphic.

(iii) $\Rightarrow$ (i). Assume $f_{i'i} : F_i \to F_{i'}$ injective for $i \leq i'$. Then, for any object $X = \text{“}\varinjlim_{j\in J}\text{”}\, X_j$ of $\mathcal{I}nd(\mathcal{S}ns)$, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(X, \mathrm{S}(\mathrm{L}(F))) &\simeq \operatorname{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(X, \mathrm{S}(\varinjlim_{i\in I} F_i)) \\
&\simeq \operatorname{Hom}_{\mathcal{B}c}(\mathrm{L}(X), \varinjlim_{i\in I} F_i) \\
&\simeq \varprojlim_{j\in J} \operatorname{Hom}_{\mathcal{B}c}(X_j, \varinjlim_{i\in I} F_i) \\
&\simeq \varprojlim_{j\in J} \varinjlim_{i\in I} \operatorname{Hom}_{\mathcal{B}c}(X_j, F_i) \qquad (*) \\
&\simeq \operatorname{Hom}_{\mathcal{I}nd(\mathcal{S}ns)}(X, F)
\end{aligned}$$

where the isomorphism (*) follows from Proposition 2.10. Since these isomorphisms are functorial in X, we get $(\mathrm{S} \circ \mathrm{L})(F) \simeq F$. □

Corollary 3.5. *The category $\mathcal{B}c$ is equivalent to the strictly full subcategory of $\mathcal{I}nd(\mathcal{S}ns)$ formed by essentially monomorphic objects.*

Remark 3.6. Let E be an infinite dimensional Banach space and let $(e_i)_{i\in\mathbb{N}}$ a free sequence of E. For any $n \in \mathbb{N}$, the linear hull $\rangle e_0, \cdots, e_n\langle$ is a finite dimensional subspace of E. It is thus closed and

$$E_n = E / \rangle e_0, \cdots, e_n\langle$$

is a Banach space. For $n \leq m$, we get a canonical morphism

$$e_{mn} : E_n \to E_m$$

and it is clear that

$$e_{pm} \circ e_{mn} = e_{pn}$$

for any $n \leq m \leq p$. Moreover since

$$\begin{aligned}
\operatorname{Ker} e_{mn} &= \rangle e_0, \cdots, e_m\langle / \rangle e_0, \cdots, e_n\langle \\
&\simeq \rangle e_{n+1}, \cdots, e_m\langle
\end{aligned}$$

we have $\dim \operatorname{Ker} e_{mn} = m - n$. Therefore for $n \in \mathbb{N}$ fixed, the sequence

$$(\operatorname{Ker} e_{mn})_{m\geq n}$$

is *not* stationary and the ind-object $\text{“}\varinjlim_{n\in\mathbb{N}}\text{”}\, E_n$ is not essentially monomorphic. This shows that S is not an equivalence of categories contrary to what is stated in [1].

Proposition 3.7. *The functor*

$$\mathrm{S} : \mathcal{B}c \to \mathcal{I}nd(\mathcal{S}ns)$$

is strictly exact (i.e. transforms any strictly exact sequence

$$E \xrightarrow{u} F \xrightarrow{v} G$$

of $\mathcal{B}c$ into the strictly exact sequence

$$\mathrm{S}(E) \xrightarrow{\mathrm{S}(u)} \mathrm{S}(F) \xrightarrow{\mathrm{S}(v)} \mathrm{S}(G)$$

of $\mathcal{I}nd(\mathcal{S}ns)$).

Proof. Since S has a left adjoint, it is kernel preserving. It is thus sufficient to show that S is exact. Let

$$0 \to E \xrightarrow{u} F \xrightarrow{v} G \to 0$$

be a strictly exact sequence of $\mathcal{B}c$ and let B be an absolutely convex bounded subset of F. Clearly $u^{-1}(B)$ (resp. $v(B)$) is an absolutely convex bounded subset of E (resp. G). Moreover, one checks easily that the sequence

$$0 \to E_{u^{-1}(B)} \to F_B \to G_{v(B)} \to 0$$

is strictly exact in $\mathcal{S}ns$. The functor $\underset{B \in \tilde{\mathcal{B}}_F}{\text{“}\varinjlim\text{”}}$ being exact, the sequence

$$0 \to \underset{B \in \tilde{\mathcal{B}}_F}{\text{“}\varinjlim\text{”}} E_{u^{-1}(B)} \to \underset{B \in \tilde{\mathcal{B}}_F}{\text{“}\varinjlim\text{”}} F_B \to \underset{B \in \tilde{\mathcal{B}}_F}{\text{“}\varinjlim\text{”}} G_{v(B)} \to 0$$

is thus strictly exact in $\mathcal{I}nd(\mathcal{S}ns)$. Since the inclusions

$$\{u^{-1}(B) : B \in \tilde{\mathcal{B}}_F\} \subset \tilde{\mathcal{B}}_E \qquad \text{and} \qquad \{v(B) : B \in \tilde{\mathcal{B}}_F\} \subset \tilde{\mathcal{B}}_G$$

are cofinal, the sequence

$$0 \to \underset{B \in \tilde{\mathcal{B}}_E}{\text{“}\varinjlim\text{”}} E_B \to \underset{B \in \tilde{\mathcal{B}}_F}{\text{“}\varinjlim\text{”}} F_B \to \underset{B \in \tilde{\mathcal{B}}_G}{\text{“}\varinjlim\text{”}} G_B \to 0$$

is strictly exact in $\mathcal{I}nd(\mathcal{S}ns)$ and the conclusion follows. □

Lemma 3.8. *Let I be a small filtering preordered set and let E be an object of $\mathcal{B}c^I$. Assume E_i is a semi-normed space for any $i \in I$. Then, the following conditions are equivalent:*

(i) E satisfies condition SC';

(ii) $\text{“}\varinjlim_{i\in I}\text{”}\, E_i$ *is essentially monomorphic.*

Proof. (i) ⇒ (ii). Let $i \in I$ and let b_i be the unit ball of E_i. There is $j \geq i$ such that

$$b_i \cap r_i^{-1}(0) \subset e_{ji}^{-1}(0).$$

Using the linearity of r_i and e_{ji} and the fact that

$$\bigcup_{r>0} rb_i = E_i,$$

we get that $\operatorname{Ker} r_i \subset \operatorname{Ker} e_{ji}$ and the conclusion follows from Proposition A.2.

(ii) ⇒ (i). This follows from Proposition A.2. □

Proposition 3.9. *The functor*

$$\mathrm{L} : \mathcal{I}nd(\mathcal{S}ns) \to \mathcal{B}c$$

is cokernel preserving and left derivable. Moreover, its cohomological dimension with respect to the right t-structures is equal to 1*. In particular, we get functors*

$$\mathrm{LL} : D^*(\mathcal{I}nd(\mathcal{S}ns)) \to D^*(\mathcal{B}c) \qquad (* \in \{-, +, b, \emptyset\})$$

Proof. Since L has a right adjoint, it is cokernel preserving. We may factor L as

$$\mathcal{I}nd(\mathcal{S}ns) \xrightarrow{\mathcal{I}nd(I)} \mathcal{I}nd(\mathcal{B}c) \xrightarrow{\mathrm{L}_{\mathcal{B}c}} \mathcal{B}c$$

where $I : \mathcal{S}ns \to \mathcal{B}c$ is the canonical inclusion functor. The first functor is kernel and cokernel preserving and since direct sums are exact in $\mathcal{B}c$, the second one is left derivable. Therefore, the functor L is left derivable if there are enough objects in $\mathcal{I}nd(\mathcal{S}ns)$ whose image in $\mathcal{I}nd(\mathcal{B}c)$ are acyclic for $\mathrm{L}_{\mathcal{B}c}$. Obviously the objects of the form $\bigoplus_{i\in I} \text{“}E_i\text{”}$ $(E_i \in \mathcal{S}ns)$ are of the requested type and we have

$$\mathrm{LL} \simeq \mathrm{LL}_{\mathcal{B}c} \circ \mathcal{I}nd(I).$$

If $E = \text{“}\varinjlim_{i\in I}\text{”}\, E_i$ is an object of $\mathcal{I}nd(\mathcal{S}ns)$, we get

$$\mathrm{LL}(E) \simeq \mathrm{LL}_{\mathcal{B}c}(\text{“}\varinjlim_{i\in I}\text{”}\, I(E_i)) \simeq \mathrm{L}\varinjlim_{i\in I} I(E_i).$$

Then, by Corollary 2.7, for $k \geq 2$, we have

$$RH_k(\mathrm{LL}(E)) \simeq RH_k(\mathrm{L}\varinjlim_{i\in I} I(E_i)) = 0.$$

Therefore, the cohomological dimension of L with respect to the right t-structure is not greater than 1. The fact that it is equal to 1 comes from Proposition 2.6, Lemma 3.8 and Remark 3.6. □

Proposition 3.10. *The functors*

$$\mathrm{S} : D(\mathcal{B}c) \to D(\mathcal{I}nd(\mathcal{S}ns))$$

$$\mathrm{LL} : D(\mathcal{I}nd(\mathcal{S}ns)) \to D(\mathcal{B}c)$$

are quasi-inverse equivalence of categories which induce the equivalence of categories

$$\mathcal{LH}(\mathcal{B}c) \approx \mathcal{LH}(\mathcal{I}nd(\mathcal{S}ns)).$$

Proof. First, consider an object E of $\mathcal{B}c$. By Proposition 3.4,

$$\mathrm{S}(E) \simeq \text{“}\varinjlim_{B \in \tilde{\mathcal{B}}_E}\text{”}\, E_B$$

is essentially monomorphic and by Lemma 3.8, the inductive system $(E_B)_{B \in \tilde{\mathcal{B}}_E} \in \mathcal{S}ns^{\tilde{\mathcal{B}}_E}$ satisfies condition SC'. It follows from Proposition 2.6 that $(E_B)_{B \in \tilde{\mathcal{B}}_E}$ is $\varinjlim\limits_{B \in \tilde{\mathcal{B}}_E}$-acyclic. Then, we get successively

$$\mathrm{LL} \circ \mathrm{S}(E) \simeq \mathrm{L} \varinjlim_{B \in \tilde{\mathcal{B}}_E} E_B \simeq \varinjlim_{B \in \tilde{\mathcal{B}}_E} E_B \simeq E$$

where the last isomorphism follows from Proposition 2.9.

Next, let E be an object of $\mathcal{I}nd(\mathcal{S}ns)$. We know that E has a resolution by L-acyclic objects of the form $\bigoplus_{i \in I} \text{“}E_i\text{”}$. Since

$$\bigoplus_{i \in I} \text{“}E\text{”}_i \simeq \varinjlim_{J \in \mathcal{P}_f(I)} \bigoplus_{j \in J} \text{“}E_j\text{”} \simeq \text{“}\varinjlim_{J \in \mathcal{P}_f(I)}\text{”} \bigoplus_{j \in J} E_j,$$

$\bigoplus_{i \in I} \text{“}E\text{”}_i$ is an essentially monomorphic ind-object. By Proposition 3.4, we get

$$\mathrm{S}(\mathrm{L}(\bigoplus_{i \in I} \text{“}E_i\text{”})) \simeq \bigoplus_{i \in I} \text{“}E_i\text{”}.$$

It follows that

$$\mathrm{S} \circ \mathrm{LL} \simeq \mathrm{id}\,.$$

To conclude, we have only to note that since

$$\mathrm{S} : \mathcal{B}c \to \mathcal{I}nd(\mathcal{S}ns)$$

is strictly exact,

$$\mathrm{S} : D(\mathcal{B}c) \to D(\mathcal{I}nd(\mathcal{S}ns))$$

is exact for the left t-structures. □

4 The category $\widehat{\mathcal{B}c}$

Definition 4.1. Let E be an object of $\mathcal{B}c$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges to a limit* $x \in E$ if there is an absolutely convex bounded subset B of E such that the sequence

$$x_n \to x \text{ in } E_B.$$

The object E is *separated* if a sequence which converges in E has a unique limit. We denote $\widehat{\mathcal{B}c}$ the full subcategory of $\mathcal{B}c$ formed by separated objects.

Remark 4.2.

(a) If $f : E \to F$ is a morphism of $\mathcal{B}c$ and $x_n \to x$ in E, then $f(x_n) \to f(x)$ in F.

(b) The category $\mathcal{N}vs$ of normed vector spaces is a full subcategory of $\widehat{\mathcal{B}c}$.

(c) An object E of $\mathcal{B}c$ is separated if and only if E_B is a normed space for any absolutely convex bounded subset B of E.

Definition 4.3. Let E be an object of $\mathcal{B}c$. A subspace F of E is *closed* if limits of sequences $(x_n)_{n\in\mathbb{N}}$ of F which converge in E belong to F. The *closure* $\overline{F}$ of a subspace F of E is the intersection of all the closed subspaces of E containing F. It is of course a closed subspace of E.

Remark 4.4.

(a) A subspace F of E is closed if and only if $F \cap E_B$ is closed in E_B for any absolutely convex bounded subset B of E.

(b) The closure of a subspace F of E is *not* always equal to the subspace formed by the limits of the sequences of F which converge in E.

(c) An object E of $\mathcal{B}c$ is separated if and only if $\{0\}$ is closed in E.

(d) A subspace F of an object E of $\widehat{\mathcal{B}c}$ is closed if and only if $E/F$is separated.

The following result is well-known.

Proposition 4.5. *Let $(E_i)_{i\in I}$ be a family of objects of $\mathcal{B}c$. Assume each E_i is separated. Then, both $\bigoplus_{i\in I} E_i$ and $\prod_{i\in I} E_i$ are separated. In particular, they form the direct sum and direct product of the family $(E_i)_{i\in I}$ in $\widehat{\mathcal{B}c}$.*

Let us now study $\widehat{\mathcal{B}c}$ from the point if view of [5].

Proposition 4.6. *Let $u : E \to F$ be a morphism of $\widehat{\mathcal{B}c}$. Then in $\widehat{\mathcal{B}c}$,* *(a)* $\operatorname{Ker} u$ *is the subspace* $u^{-1}(0)$ *of* E *with the bornology induced by that of* E*,*

(b) $\operatorname{Coker} u$ *is the quotient space* $F/\overline{u(E)}$ *with the quotient bornology,*

(c) $\operatorname{Im} u$ *is the subspace* $\overline{u(E)}$ *of* F *with the bornology induced by that of* F*,*

(d) $\operatorname{Coim} u$ *is the quotient space* $E/u^{-1}(0)$ *of* E *with the quotient bornology,*

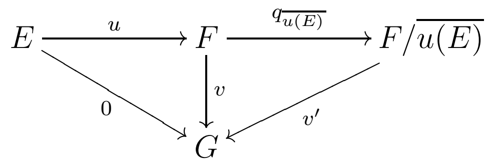
(e) u *is strict if and only if* u *is strict in* $\mathcal{B}c$ *and* $u(E)$ *is a closed subspace of* F*.*

Proof. (a) We know that $u^{-1}(0)$ is the kernel of u in $\mathcal{B}c$. Since $u^{-1}(0)$ is endowed with the induced bornology, $u^{-1}(0)$ is separated. Hence, $u^{-1}(0)$ is the kernel of u in $\widehat{\mathcal{B}}c$.

(b) Since $\overline{u(E)}$ is closed, $F/\overline{u(E)}$ is separated. Let $v : F \to G$ be a morphism of $\widehat{\mathcal{B}}c$ such that $v \circ u = 0$. Since $v^{-1}(0)$ is a closed subspace of F, $\overline{u(E)} \subset v^{-1}(0)$. Then, the linear map

$$v' : F/\overline{u(E)} \to G \qquad [f]_{\overline{u(E)}} \mapsto v(f)$$

is well-defined and is clearly a morphism of $\widehat{\mathcal{B}}c$. Since v' is the unique morphism making the diagram



commutative, we see that $F/\overline{u(E)}$ is the cokernel of u in $\widehat{\mathcal{B}}c$.

(c) follows from (a) and (b).

(d) We know that $E/u^{-1}(0)$ is the coimage of u in $\mathcal{B}c$. Since $\{0\}$ is closed in F, $u^{-1}(0)$ is closed in E and $E/u^{-1}(0)$ is separated. Hence, $E/u^{-1}(0)$ is the coimage of u in $\widehat{\mathcal{B}}c$.

(e) Assume that u is strict in $\widehat{\mathcal{B}}c$. This means that the canonical morphism

$$E/u^{-1}(0) \to \overline{u(E)}$$

is an isomorphism of separated spaces. The linear map

$$E/u^{-1}(0) \to u(E)$$

being bijective, it follows that $u(E) = \overline{u(E)}$ and that u is strict in $\mathcal{B}c$. The converse is obtained by reversing the preceding arguments. □

Corollary 4.7. *Let $u : E \to F$ be a morphism of $\widehat{\mathcal{B}c}$. Then,*

(i) u is a strict epimorphism of $\widehat{\mathcal{B}c}$ if and only if u is a strict epimorphism of $\mathcal{B}c$;

(ii) u is a strict monomorphism of $\widehat{\mathcal{B}c}$ if and only if u is a strict monomorphism of $\mathcal{B}c$ which has a closed range.

Lemma 4.8. *A sequence*

$$E \xrightarrow{u} F \xrightarrow{v} G$$

of $\widehat{\mathcal{B}c}$ is strictly exact if and only if it is strictly exact in $\mathcal{B}c$.

Proof. Recall first that a null sequence

$$E \xrightarrow{u} F \xrightarrow{v} G$$

in an additive category with kernels and cokernels is strictly exact if and only if u is strict and $\operatorname{Im} u = \operatorname{Ker} v$.

Therefore, if the sequence of the statement is strictly exact in $\widehat{\mathcal{B}c}$, it follows from Proposition 4.6 that u is a strict morphism of $\mathcal{B}c$ which has a closed range and that $\overline{u(E)} = v^{-1}(0)$. In particular, $u(E) = v^{-1}(0)$ and the sequence is strictly exact in $\mathcal{B}c$.

Conversely, assume that the sequence of the statement is strictly exact in $\mathcal{B}c$. In this case, u is a strict morphism of $\mathcal{B}c$ and $u(E) = v^{-1}(0)$. Since G is separated, $v^{-1}(0)$ is closed in F. Hence, $u(E)$ is closed in F and u is strict in $\widehat{\mathcal{B}c}$. The conclusion follows. □

Lemma 4.9. *Let $u : E \to F$ and $v : F \to G$ be two morphisms of $\widehat{\mathcal{B}c}$. Assume $w = v \circ u$ is a strict monomorphism of $\widehat{\mathcal{B}c}$, then u is a strict monomorphism of $\widehat{\mathcal{B}c}$.*

Proof. By Corollary 4.7, $v \circ u$ is a strict monomorphism of $\mathcal{B}c$. Since $\mathcal{B}c$ is quasi-abelian, it follows that u is a strict monomorphism of $\mathcal{B}c$. So, we only have to prove that u has a closed range. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of E such that $u(x_n) \to y$ in F. Clearly, $v(u(x_n)) \to v(y)$ in G and since w is a strict monomorphism of $\mathcal{B}c$ with closed range, there is an x in E with $w(x) = v(y)$ and for which $x_n \to x$. Since u is a morphism of $\mathcal{B}c$, $u(x_n) \to u(x)$ in F, and F being separated, we get $u(x) = y$. Hence, $y \in u(E)$ and the conclusion follows. □

Proposition 4.10. *The category $\widehat{\mathcal{B}c}$ is quasi-abelian.*

Proof. We know that $\widehat{\mathcal{B}c}$ is additive and that any morphism of $\widehat{\mathcal{B}c}$ has a kernel and a cokernel.

Consider a cartesian square

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle v}\uparrow & & \uparrow{\scriptstyle g} \\ T & \xrightarrow[u]{} & G \end{array}$$

of $\widehat{\mathcal{B}}c$ where f is a strict epimorphism. By Corollary 4.7, f is a strict epimorphism of $\mathcal{B}c$. It follows that u is a strict epimorphism of $\mathcal{B}c$ and hence of $\widehat{\mathcal{B}}c$.

Consider now a cocartesian square

$$\begin{array}{ccc} G & \xrightarrow{u} & T \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle v} \\ E & \xrightarrow[f]{} & F \end{array} \qquad (*)$$

of $\widehat{\mathcal{B}}c$ where f is a strict monomorphism. Since the diagram

$$\begin{array}{ccc} E & \xrightarrow{\left(\begin{smallmatrix} g \\ -f \end{smallmatrix}\right)} & G \oplus F \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle (0\ -1)} \\ & & F \end{array}$$

is commutative, by Lemma 4.9,

$$\left(\begin{smallmatrix} g \\ -f \end{smallmatrix}\right) : E \to G \oplus F$$

is a strict monomorphism of $\widehat{\mathcal{B}}c$. The square (*) being cocartesian, it follows that the sequence

$$0 \longrightarrow E \xrightarrow{\left(\begin{smallmatrix} g \\ -f \end{smallmatrix}\right)} G \oplus F \xrightarrow{(u\ v)} T \longrightarrow 0 \qquad (**)$$

is strictly exact in $\widehat{\mathcal{B}}c$ and hence in $\mathcal{B}c$ by Lemma 4.8. The square (*) is thus cocartesian in $\mathcal{B}c$ and u is a strict monomorphism of $\mathcal{B}c$. To conclude, it remains to prove that u has closed range. The morphisms g and v induce a strict quasi-isomorphism between the complexes

$$0 \to E \xrightarrow{f} F \to 0 \qquad \text{and} \qquad 0 \to G \xrightarrow{u} T \to 0$$

since the mapping cone of

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \xrightarrow{f} & F & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & G & \xrightarrow[u]{} & T & \longrightarrow & 0 \end{array}$$

is the strictly exact complex (**). Taking the cohomology and using the fact that f and u are strict monomorphisms of $\mathcal{B}c$, we get

$$\operatorname{Coker}(f) \simeq \operatorname{Coker}(u) \qquad \text{in } \mathcal{B}c.$$

Since $f(E)$ is closed, $\operatorname{Coker}(f) = F/f(E)$ is separated. It follows that $\operatorname{Coker}(u) = T/u(G)$ is separated and hence that $u(G)$ is closed. □

Proposition 4.11.

(a) *The projective objects of $\mathcal{B}c$ are separated.*

(b) *The category $\widehat{\mathcal{B}c}$ has enough projective objects.*

Proof.
(a) It follows from the proof of Proposition 2.13 that any projective object of $\mathcal{B}c$ is a direct summand of an object of the type

$$\bigoplus_{i \in I} P_i$$

where each P_i is a projective object of $\mathcal{S}ns$. So we are reduced to prove that the projective objects of $\mathcal{S}ns$ are separated. This follows easily from the fact these objects are direct summands of normed spaces of the type

$$\smile\!\!\!\!\!\!\bigoplus_{j \in J} \mathbb{C}$$

(see [5, Proposition 3.2.12]).

(b) Thanks to Proposition 2.13 and Corollary 4.7, this is a direct consequence of (a).

□

Proposition 4.12. *The category $\widehat{\mathcal{B}c}$ is complete and cocomplete. Moreover, direct sums are kernel and cokernel preserving and direct products are strictly exact.*

Proof. The result follows from Proposition 1.9 thanks to Proposition 4.5 and Proposition 4.8. □

Definition 4.13. We denote by

$$\widehat{\mathrm{I}} : \widehat{\mathcal{B}c} \to \mathcal{B}c$$

the inclusion functor and we define the functor

$$\widehat{\mathrm{Sep}} : \mathcal{B}c \to \widehat{\mathcal{B}c}$$

by setting

$$\widehat{\mathrm{S}}\mathrm{ep}(E) = E/\overline{\{0\}}^E$$

and endowing it with the quotient bornology.

One checks easily that:

Proposition 4.14. *For any object E of $\mathcal{B}c$ and any object F of $\widehat{\mathcal{B}}c$, we have the adjunction formula*

$$\mathrm{Hom}_{\widehat{\mathcal{B}}c}(\widehat{\mathrm{S}}\mathrm{ep}(E), F) \simeq \mathrm{Hom}_{\mathcal{B}c}(E, \widehat{\mathrm{I}}(F)).$$

Moreover,

$$\widehat{\mathrm{S}}\mathrm{ep} \circ \widehat{\mathrm{I}} = \mathrm{id}_{\widehat{\mathcal{B}}c}.$$

Proposition 4.15. *Any object E of $\mathcal{B}c$ has a presentation of the form*

$$0 \to S_1 \to S_0 \to E \to 0$$

where S_1, S_0 are objects of $\widehat{\mathcal{B}}c$.

Proof. By Proposition 2.13 $\mathcal{B}c$ has enough projective objects and by Proposition 4.11 these objects are separated. It follows that for any E in $\mathcal{B}c$ there is a strict epimorphism

$$u : S_0 \to E$$

with S_0 separated. Taking $S_1 = \mathrm{Ker}\, u$ allows us to conclude. □

Proposition 4.16. *(a) The functor*

$$\widehat{\mathrm{S}}\mathrm{ep} : \mathcal{B}c \to \widehat{\mathcal{B}}c$$

is cokernel preserving, left derivable and has bounded cohomological dimension. In particular it has a left derived functor

$$\mathrm{L}\widehat{\mathrm{S}}\mathrm{ep} : D^*(\mathcal{B}c) \to D^*(\widehat{\mathcal{B}}c) \qquad * \in \{\emptyset, +, -, b\}.$$

(b) The functor

$$\widehat{\mathrm{I}} : \widehat{\mathcal{B}}c \to \mathcal{B}c$$

is strictly exact and gives rise to a functor

$$\widehat{\mathrm{I}} : D^*(\widehat{\mathcal{B}}c) \to D^*(\mathcal{B}c) \qquad * \in \{\emptyset, +, -, b\}.$$

(c) Moreover, $\widehat{\mathrm{I}}$ and $\mathrm{L}\widehat{\mathrm{S}}\mathrm{ep}$ define quasi-inverse equivalence of categories. In particular,

$$\widehat{\mathrm{I}} : \mathcal{LH}(\widehat{\mathcal{B}}c) \to \mathcal{LH}(\mathcal{B}c)$$

is an equivalence of categories.

Proof. (a) Since $\widehat{\mathrm{I}}$ is a right adjoint of $\widehat{\mathrm{S}}\mathrm{ep}$, $\widehat{\mathrm{S}}\mathrm{ep}$ is cokernel preserving. It follows from Proposition 4.15 and Lemma 4.8 that $\widehat{\mathcal{B}c}$ forms a $\widehat{\mathrm{S}}\mathrm{ep}$-projective subcategory of $\mathcal{B}c$. Hence, the functor $\widehat{\mathrm{S}}\mathrm{ep}$ is left derivable and has bounded cohomological dimension.

(b) This follows directly from Lemma 4.8.

(c) On one hand, for any object S of $D(\widehat{\mathcal{B}c})$, we have

$$\mathrm{L}\widehat{\mathrm{S}}\mathrm{ep} \circ \widehat{\mathrm{I}}(S) = \mathrm{L}\widehat{\mathrm{S}}\mathrm{ep}(\widehat{\mathrm{I}}(S)) = \widehat{\mathrm{S}}\mathrm{ep}(\widehat{\mathrm{I}}(S)) = S$$

where the second equality follows from the fact that the components of the complex $\widehat{\mathrm{I}}(S)$ are in a $\widehat{\mathrm{S}}\mathrm{ep}$-projective subcategory of $\mathcal{B}c$. On the other hand, any object E of $D(\mathcal{B}c)$ is quasi-isomorphic to a complex S with separated components. Therefore, we have

$$\widehat{\mathrm{I}} \circ \mathrm{L}\widehat{\mathrm{S}}\mathrm{ep}(E) \simeq \widehat{\mathrm{I}} \circ \widehat{\mathrm{S}}\mathrm{ep}(S) \simeq S \simeq E.$$

To conclude, it is sufficient to note that, since

$$\widehat{\mathrm{I}} : \widehat{\mathcal{B}c} \to \mathcal{B}c$$

is strictly exact, the functor

$$\widehat{\mathrm{I}} : D(\widehat{\mathcal{B}c}) \to D(\mathcal{B}c)$$

is exact for the left t-structures. □

Proposition 4.17. *Let I be a filtering preordered set and let E be an object of $\mathcal{B}c^I$. Assume that E satisfies condition SC' and that E_i is separated for any $i \in I$. Then $\varinjlim_{i\in I} E_i$ is separated.*

Proof. Set $L = \varinjlim_{i\in I} E_i$ and let B be an absolutely convex bounded subset of L. We know that there is $i \in I$ and an absolutely convex bounded subset B_i of E_i such that

$$B = r_i(B_i).$$

Proceeding as in the proof of Proposition 2.10, we may even assume that

$$r_i : (E_i)_{B_i} \to L$$

is injective. It follows that $L_B \simeq (E_i)_{B_i}$ as semi-normed spaces. By assumption, $(E_i)_{B_i}$ is in fact a normed space. Hence, L_B is also a normed space and the conclusion follows. □

Proposition 4.18. *Let S be a filtering inductive system of $\widehat{\mathcal{B}c}$ indexed by J. Then,*

$$\widehat{\mathrm{I}}(\mathrm{L}\varinjlim_{j\in J} S_j) \simeq \mathrm{L}\varinjlim_{j\in J} \widehat{\mathrm{I}}(S_j).$$

Proof. Since S has a left resolution by inductive systems of coproduct type, the result will be true if it is true when S is itself of coproduct type. In this case, S and $\widehat{\mathrm{I}} \circ S$ are both $\mathrm{L}\varinjlim_{j\in J}$-acyclic. Since S has also injective transitions the conclusion follows from Proposition 4.17. □

Proposition 4.19.

(a) *The functor*

$$\widehat{\mathrm{S}} : \widehat{\mathcal{B}c} \to \mathcal{I}nd(\mathcal{N}vs)$$
$$E \mapsto \text{“}\varinjlim_{B\in\tilde{\mathcal{B}}_E}\text{”}\, E_B$$

has as left adjoint the functor

$$\widehat{\mathrm{L}} : \mathcal{I}nd(\mathcal{N}vs) \to \widehat{\mathcal{B}c}$$
$$\text{“}\varinjlim_{i\in I}\text{”}\, E_i \mapsto \varinjlim_{i\in I} E_i$$

where the limit is taken in $\widehat{\mathcal{B}c}$ (and may differ from the corresponding limit in $\mathcal{B}c$).

(b) *The canonical morphism*

$$\widehat{\mathrm{L}} \circ \widehat{\mathrm{S}}(E) \to E$$

is an isomorphism for any E in $\widehat{\mathcal{B}c}$.

(c) *For any F in $\mathcal{I}nd(\mathcal{N}vs)$, the following conditions are equivalent:*

(i) *the canonical morphism $F \to \widehat{\mathrm{S}} \circ \widehat{\mathrm{L}}(F)$ is an isomorphism;*

(ii) *F is in the essential image of $\widehat{\mathrm{S}}$;*

(iii) *F is essentially monomorphic.*

Proof. Thanks to Proposition 4.17, we may proceed entirely as in the proof of Proposition 3.4. □

Corollary 4.20. *The category $\widehat{\mathcal{B}c}$ is equivalent to the full subcategory of $\mathcal{I}nd(\mathcal{N}vs)$ formed by essentially monomorphic objects.*

Proposition 4.21. *(a) The functor*

$$\widehat{\mathrm{S}} : \widehat{\mathcal{B}c} \to \mathcal{I}nd(\mathcal{N}vs)$$

is strictly exact and has bounded cohomological dimension. In particular, it induces a functor

$$\widehat{\mathrm{S}} : D^*(\widehat{\mathcal{B}c}) \to D^*(\mathcal{I}nd(\mathcal{N}vs)) \qquad (* \in \{-,+,b,\emptyset\}).$$

(b) The functor

$$\widehat{\mathrm{L}} : \mathcal{I}nd(\mathcal{N}vs) \to \widehat{\mathcal{B}c}$$

is cokernel preserving and has a left derived functor

$$\mathrm{L}\widehat{\mathrm{L}} : D^*(\mathcal{I}nd(\mathcal{N}vs)) \to D^*(\widehat{\mathcal{B}c}).$$

(c) The functors $\widehat{\mathrm{S}}$ and $\mathrm{L}\widehat{\mathrm{L}}$ are quasi-inverse equivalence of categories. They induce an equivalence of categories

$$\mathcal{LH}(\widehat{\mathcal{B}c}) \simeq \mathcal{LH}(\mathcal{I}nd(\mathcal{N}vs)).$$

Proof.

(a) Thanks to Lemma 4.8, we may proceed as in Proposition 3.7.

(b) & (c) Using Proposition 3.7, and Proposition 4.18, one can adapt easily the proofs of Proposition 3.9 and Proposition 3.10. □

5 The category $\widehat{\mathcal{B}c}$

Definition 5.1. Let E be an object of $\mathcal{B}c$. We say that E is *complete* if for any bounded subset B of E, there is an absolutely convex bounded subset $B' \supset B$ of E such that $E_{B'}$ is a Banach space. We denote $\widehat{\mathcal{B}c}$ the full subcategory of $\mathcal{B}c$ formed by complete spaces. For any object E of $\widehat{\mathcal{B}c}$, we set

$$\tilde{\mathcal{B}}'_E = \{B \in \tilde{\mathcal{B}}_E : E_B \text{ Banach}\}.$$

Remark 5.2.

(a) The category $\mathcal{B}an$ of Banach spaces is a full subcategory of $\widehat{\mathcal{B}c}$.

(b) The category $\widehat{\mathcal{B}c}$ is a full subcategory of $\widehat{\mathcal{B}c}$.

Proposition 5.3. *Let E be an object of $\mathcal{B}c$ and let F be a linear subspace of E endowed with the induced bornology. Assume E is complete and F is closed in E. Then, both F and E/F are complete.*

Proof. Let us prove that F is complete. Let B be an arbitrary bounded subset of F. By definition, there is a bounded subset B' of E such that $B = B' \cap F$. Since E is complete, there is also an absolutely convex bounded subset $B'' \supset B'$ of E such that $E_{B''}$ is a Banach space. Since F is closed in E, $E_{B''} \cap F$ is closed in $E_{B''}$. Hence,

$$F_{B''\cap F} = E_{B''} \cap F$$

is a Banach space. Since $B'' \cap F \supset B$, the conclusion follows.

Now, let us prove that E/F is complete. Denote $q : E \to E/F$ the canonical morphism and let B be a bounded subset of E/F. Since q is a strict epimorphism, by Corollary 1.7, there is a bounded subset B' of E such that $B \subset q(B')$. The space E being complete, there is an absolutely convex bounded subset $B'' \supset B'$ of E such that $E_{B''}$ is a Banach space. Therefore, $q(B'')$ is an absolutely convex bounded subset of E/F such that $B \subset q(B'')$. Moreover, we clearly have

$$(E/F)_{q(B'')} \simeq E_{B''}/F_{B''\cap F} \simeq E_{B''}/E_{B''} \cap F.$$

Since F is closed in E, $E_{B''} \cap F$ is closed in $E_{B''}$. Hence, $(E/F)_{q(B'')}$ is a Banach space and the conclusion follows. □

Remark 5.4. Let $(E_i)_{i\in I}$ be a family of semi-normed spaces and let p_i be the semi-norm of E_i. Following [5], we denote

$$\breve{\prod_{i\in I}} E_i$$

the vector subspace of $\prod_{i\in I} E_i$ formed by the families $(e_i)_{i\in I}$ such that

$$p((e_i)_{i\in I}) = \sup_{i\in I} p_i(e_i) < +\infty$$

endowed with the semi-norm p. We leave it to the reader to check that $\breve{\prod\limits_{i\in I}} E_i$ is a Banach space if E_i is a Banach space for every $i \in I$.

Proposition 5.5. *Let $(E_i)_{i\in I}$ be a family of $\mathcal{B}c$. Assume E_i is complete for $i \in I$. Then $\bigoplus_{i\in I} E_i$ and $\prod_{i\in I} E_i$ are complete. In particular, they form the direct sum and direct product of the family $(E_i)_{i\in I}$ in $\widehat{\mathcal{B}c}$.*

Proof. First, let us show that $\bigoplus_{i\in I} E_i$ is complete. Let B be a bounded subset of $\bigoplus_{i\in I} E_i$. We know that

$$B \subset \bigoplus_{i\in I} B_i$$

where B_i is a bounded subset of E_i, the set $N = \{i \in I : B_i \neq \{0\}\}$ being finite. For any $i \in N$, there is an absolutely convex bounded subset $B'_i \supset B_i$ of E_i such that $(E_i)_{B_i}$ is a Banach space. For $i \in I \setminus N$, set $B'_i = \{0\}$. Then,

$$B \subset \bigoplus_{i \in I} B'_i.$$

One checks easily that

$$\left(\bigoplus_{i \in I} E_i\right)_{\bigoplus_{i \in I} B'_i} \simeq \bigoplus_{i \in N} (E_i)_{B'_i}.$$

Since a finite direct sum of Banach spaces is a Banach space, these spaces are Banach spaces and the conclusion follows.

Next, let us prove that $\prod_{i \in I} E_i$ is complete. Let B be a bounded subset of $\prod_{i \in I} E_i$. We know that

$$B \subset \prod_{i \in I} B_i$$

where B_i is a bounded subset of E_i for any $i \in I$. For each $i \in I$, one can find an absolutely convex bounded subset B'_i of E_i containing B_i for which $(E_i)_{B'_i}$ is a Banach space. A simple computation shows that

$$\left(\prod_{i \in I} E_i\right)_{\prod_{i \in I} B'_i} \simeq \overset{\smile}{\prod_{i \in I}} (E_i)_{B'_i}.$$

and the conclusion follows from Remark 5.4. □

Proposition 5.6.

(a) *The kernel, cokernel, image and coimage of a morphism*

$$u : E \to F$$

of $\widehat{\mathcal{B}c}$ are the same as those obtained by considering u as a morphism of $\widehat{\mathcal{B}}c$. Moreover, u is strict in $\widehat{\mathcal{B}c}$ if and only if it is strict in $\widehat{\mathcal{B}}c$.

(b) *The category $\widehat{\mathcal{B}c}$ is quasi-abelian.*

(c) *The category $\widehat{\mathcal{B}c}$ is complete and co-complete. Moreover, direct sums are kernel and cokernel preserving and direct products are strictly exact.*

Proof. (a) We know that $u^{-1}(0)$ is the kernel of u in $\widehat{\mathcal{B}}c$ and that it is a closed subspace of E. It follows that $u^{-1}(0)$ is complete and hence that it is the kernel of u in $\widehat{\mathcal{B}c}$. Recall that the cokernel of u in $\widehat{\mathcal{B}}c$ is $F/\overline{u(E)}$. Since $\overline{u(E)}$ is closed, $F/\overline{u(E)}$ is complete. Therefore, $F/\overline{u(E)}$ is also the cokernel of u in $\widehat{\mathcal{B}c}$. The rest of part (a) is now obvious.

(b) This follows from (a) combined with the fact that the category $\widehat{\mathcal{B}}c$ is quasi-abelian.

(c) Thanks to (a), (b) and the preceding proposition, this is an easy consequence of the similar result for $\widehat{\mathcal{B}}c$. □

Corollary 5.7.

(i) A sequence

$$E \xrightarrow{u} F \xrightarrow{v} G$$

of $\widehat{\mathcal{B}c}$ is strictly exact if and only if it is strictly exact in $\mathcal{B}c$.

(ii) If $(E_i)_{i\in I}$ is an inductive system of $\widehat{\mathcal{B}c}$, then its inductive limit in $\widehat{\mathcal{B}c}$ is equal to its inductive limit in $\widehat{\mathcal{B}}c$ (but not *in general in $\mathcal{B}c$).*

Proof.

(i) follows directly from Lemma 4.8 and Proposition 5.6.

(ii) Since direct sums and cokernels are the same in $\widehat{\mathcal{B}}c$ and $\widehat{\mathcal{B}c}$, the sequence

$$\bigoplus_{i\leq j} E_i \to \bigoplus_{i\in I} E_i \to \varinjlim_{i\in I} E_i \to 0$$

is both strictly coexact in $\widehat{\mathcal{B}}c$ or $\widehat{\mathcal{B}c}$. The conclusion follows. □

Proposition 5.8. *The category $\widehat{\mathcal{B}c}$ has enough projective objects.*

Proof. Let E be an object of $\widehat{\mathcal{B}c}$. The inclusion $\tilde{\mathcal{B}}'_E \subset \tilde{\mathcal{B}}_E$ being cofinal, the canonical morphism

$$\bigoplus_{B\in\tilde{\mathcal{B}}'_E} E_B \to E$$

is a strict epimorphism. Since $\mathcal{B}an$ has enough projective objects, we can conclude by proceeding as for Proposition 2.13. □

Lemma 5.9. *Let E be a normed space and let F be an object of $\widehat{\mathcal{B}c}$. Then,*

$$\mathrm{Hom}_{\widehat{\mathcal{B}c}}(\widehat{E}, F) \simeq \mathrm{Hom}_{\mathcal{B}c}(E, F).$$

Proof. Using Proposition 2.9, Proposition 2.10 and the fact that the inclusion $\tilde{\mathcal{B}}'_F \subset \tilde{\mathcal{B}}_F$ is cofinal, we get successively

$$\begin{aligned}\operatorname{Hom}_{\mathcal{B}c}(E,F) &\simeq \operatorname{Hom}_{\mathcal{B}c}(E, \varinjlim_{B\in\tilde{\mathcal{B}}'_F} F_B)\\ &\simeq \varinjlim_{B\in\tilde{\mathcal{B}}'_F} \operatorname{Hom}_{\mathcal{B}an}(E,F_B)\\ &\simeq \varinjlim_{B\in\tilde{\mathcal{B}}'_F} \operatorname{Hom}_{\mathcal{B}an}(\widehat{E},F_B)\\ &\simeq \operatorname{Hom}_{\mathcal{B}c}(\widehat{E}, \varinjlim_{B\in\tilde{\mathcal{B}}'_F} F_B)\\ &\simeq \operatorname{Hom}_{\widehat{\mathcal{B}c}}(\widehat{E},F).\end{aligned}$$

□

Definition 5.10. We denote by

$$\widehat{\mathrm{I}} : \widehat{\mathcal{B}c} \to \mathcal{B}c$$

the canonical inclusion functor and we define the functor

$$\widehat{\mathrm{Cpl}} : \mathcal{B}c \to \widehat{\mathcal{B}c}$$

by setting

$$\widehat{\mathrm{Cpl}}(E) = \varinjlim_{B\in\tilde{\mathcal{B}}_E} \widehat{E}_B$$

where the inductive limit is taken in $\widehat{\mathcal{B}c}$ (and may differ from the inductive limit in $\mathcal{B}c$).

Proposition 5.11. *For any object E of $\mathcal{B}c$ and any object F of $\widehat{\mathcal{B}c}$, we have the adjunction formula:*

$$\operatorname{Hom}_{\widehat{\mathcal{B}c}}(\widehat{\mathrm{Cpl}}(E),F) \simeq \operatorname{Hom}_{\mathcal{B}c}(E,\widehat{\mathrm{I}}(F))$$

Moreover,

$$\widehat{\mathrm{Cpl}} \circ \widehat{\mathrm{I}} = \mathrm{id}_{\widehat{\mathcal{B}c}}.$$

Proof. We have successively

$$\begin{aligned}
\operatorname{Hom}_{\widehat{\mathcal{B}c}}(\widehat{\mathrm{C}}\mathrm{pl}(E), F) &\simeq \operatorname{Hom}_{\widehat{\mathcal{B}c}}(\varinjlim_{B \in \tilde{\mathcal{B}}_E} \widehat{E}_B, F)\\
&\simeq \varprojlim_{B \in \tilde{\mathcal{B}}_E} \operatorname{Hom}_{\widehat{\mathcal{B}c}}(\widehat{E}_B, F)\\
&\simeq \varprojlim_{B \in \tilde{\mathcal{B}}_E} \operatorname{Hom}_{\mathcal{B}c}(E_B, F)\\
&\simeq \operatorname{Hom}_{\mathcal{B}c}(\varinjlim_{B \in \tilde{\mathcal{B}}_E} E_B, F)\\
&\simeq \operatorname{Hom}_{\mathcal{B}c}(E, \widehat{\mathrm{I}}(F)).
\end{aligned}$$

Note that the third isomorphism follows from Lemma 5.9 and that the last one is a consequence of Proposition 2.9. Moreover, for any object E of $\widehat{\mathcal{B}c}$, we have by cofinality:

$$\widehat{\mathrm{C}}\mathrm{pl} \circ \widehat{\mathrm{I}}(E) \simeq \varinjlim_{B \in \tilde{\mathcal{B}}_E} \widehat{E}_B \simeq \varinjlim_{B \in \tilde{\mathcal{B}}'_E} E_B \simeq \varinjlim_{B \in \tilde{\mathcal{B}}_E} E_B \simeq E.$$

□

Proposition 5.12. *The functor* $\widehat{\mathrm{C}}\mathrm{pl}$ *may be decomposed as* $\widehat{\mathrm{L}} \circ \mathrm{CS}$ *where*

$$\begin{array}{rccc}
\mathrm{CS}: & \mathcal{B}c & \to & \mathcal{I}nd(\mathcal{B}an)\\
& E & \mapsto & \text{“}\varinjlim_{B \in \tilde{\mathcal{B}}_E}\text{”}\, \widehat{E}_B
\end{array}$$

and

$$\begin{array}{rccc}
\widehat{\mathrm{L}}: & \mathcal{I}nd(\mathcal{B}an) & \to & \widehat{\mathcal{B}c}\\
& \text{“}\varinjlim_{i \in I}\text{”}\, E_i & \mapsto & \varinjlim_{i \in I} E_i.
\end{array}$$

Moreover, for any object E *of* $\widehat{\mathcal{B}c}$ *and any object* F *of* $\mathcal{I}nd(\mathcal{B}an)$*, we have the adjunction formula:*

$$\operatorname{Hom}_{\widehat{\mathcal{B}c}}(\widehat{\mathrm{L}}(F), E) \simeq \operatorname{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(F, \mathrm{CS}(\widehat{\mathrm{I}}(E))).$$

Proof. For any object E of $\mathcal{B}c$, we have

$$\widehat{\mathrm{L}} \circ \mathrm{CS}(E) = \widehat{\mathrm{L}}(\text{“}\varinjlim_{B \in \tilde{\mathcal{B}}_E}\text{”}\, \widehat{E}_B) = \varinjlim_{B \in \tilde{\mathcal{B}}_E} \widehat{E}_B = \widehat{\mathrm{C}}\mathrm{pl}(E).$$

Let E be an object of $\widehat{\mathcal{B}c}$ and let $F = \text{``}\varinjlim_{i\in I}\text{''}\, F_i$ be an object of $\mathcal{I}nd(\mathcal{B}an)$. Then, we have successively

$$\begin{aligned}
\operatorname{Hom}_{\widehat{\mathcal{B}c}}(\widehat{\mathrm{L}}(F), E) &\simeq \operatorname{Hom}_{\widehat{\mathcal{B}c}}(\varinjlim_{i\in I} F_i, E)\\
&\simeq \varprojlim_{i\in I} \operatorname{Hom}_{\mathcal{B}c}(F_i, E)\\
&\simeq \varprojlim_{i\in I} \varinjlim_{B\in\tilde{\mathcal{B}}_E} \operatorname{Hom}_{\mathcal{B}c}(F_i, E_B)\\
&\simeq \varprojlim_{i\in I} \varinjlim_{B'\in\tilde{\mathcal{B}}'_E} \operatorname{Hom}_{\mathcal{B}an}(F_i, E_{B'})\\
&\simeq \operatorname{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\text{``}\varinjlim_{i\in I}\text{''}\, E_i, \text{``}\varinjlim_{B'\in\tilde{\mathcal{B}}'_E}\text{''}\, E_{B'})\\
&\simeq \operatorname{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(F, \mathrm{CS}(\widehat{\mathrm{I}}(E))).
\end{aligned}$$

□

Proposition 5.13.

(a) *The functor* CS *is exact and gives rise to a functor*

$$\mathrm{CS} : D^*(\mathcal{B}c) \to D^*(\mathcal{I}nd(\mathcal{B}an)) \qquad (* \in \{-,+,b,\emptyset\}).$$

(b) *The functor* $\widehat{\mathrm{L}}$ *is cokernel preserving and has bounded cohomological dimension. In particular, it has a left derived functor*

$$\mathrm{L}\widehat{\mathrm{L}} : D^*(\mathcal{I}nd(\mathcal{B}an)) \to D^*(\widehat{\mathcal{B}c}) \qquad * \in \{-,+,b,\emptyset\}.$$

(c) *An object of* $\mathcal{I}nd(\mathcal{B}an)$ *is* $\widehat{\mathrm{L}}$*-acyclic if and only if it is essentially injective.*

Proof.

(a) Since the functor $\widehat{\cdot} : \mathcal{S}ns \to \mathcal{B}an$ is exact, the functor $\mathcal{I}nd(\widehat{\cdot})$ is also exact. By Proposition 3.7, the functor $\mathrm{S} : \mathcal{B}c \to \mathcal{I}nd(\mathcal{S}ns)$ is strictly exact. Therefore, the functor $\mathrm{CS} = \mathcal{I}nd(\widehat{\cdot}) \circ \mathrm{S}$ is exact.

(b) By the preceding proposition, $\widehat{\mathrm{L}}$ has a right adjoint; it is thus a cokernel preserving functor. Proceeding as in Proposition 3.9, we see easily that $\widehat{\mathrm{L}}$ is left derivable. Moreover, for any filtering preordered set J and any $F \in \mathcal{B}an^J$, we get

$$\mathrm{L}\widehat{\mathrm{L}}(F) \simeq \mathrm{L}\varinjlim_{j\in J} F_j$$

where the derived inductive limit is taken in $\widehat{\mathcal{B}c}$. By the theory of derived inductive limits for quasi-abelian categories, we know that

$$\mathrm{L}\varinjlim_{j\in J} F_j \simeq R.(J,F)$$

where F is considered as an object of $\widehat{\mathcal{B}c}^J$ and $R.(J,F)$ is its negative Roos complex. By Proposition 5.5,

$$\widehat{\mathrm{I}}(R.(J,F)) \simeq R.(J,\widehat{\mathrm{I}}\circ F).$$

Moreover, thanks to Proposition 3.9, we have

$$LH_k(R.(J,\widehat{\mathrm{I}}\circ F)) = 0$$

for $k \geq 1$. Applying Corollary 5.7, we deduce that

$$LH_k(R.(J,F)) = 0$$

for $k \geq 1$. In particular, $\widehat{\mathrm{L}}$ has bounded cohomological dimension and

$$\mathrm{L}\widehat{\mathrm{L}} : D^*(\mathcal{I}nd(\mathcal{B}an)) \to D^*(\widehat{\mathcal{B}c}) \qquad * \in \{-,+,b,\emptyset\}.$$

is well-defined.

(c) Let J be a filtering preorderd set and let F be an object of $\mathcal{B}an^J$.

Assume F is $\widehat{\mathrm{L}}$-acyclic. Proceeding as in (b), we see that the sequence

$$\cdots \to R_1(J,F) \to R_0(J,F) \to \widehat{\mathrm{L}}(F) \to 0$$

is strictly exact in $\widehat{\mathcal{B}c}$. It follows that the sequence

$$\cdots \to R_1(J,\widehat{\mathrm{I}}\circ F) \to R_0(J,\widehat{\mathrm{I}}\circ F) \to \widehat{\mathrm{I}}(\widehat{\mathrm{L}}(F)) \to 0$$

is strictly exact in $\mathcal{B}c$. Hence $\widehat{\mathrm{I}}\circ F$ is $\varinjlim_{j\in J}$-acyclic and by combining Proposition 2.6 and Lemma 3.8 we find that F is essentially monomorphic.

Conversely, assume F is essentially monomorphic. Then, by Lemma 3.8, F satisfies condition SC'. Using Proposition 4.17, we see that the inductive limit of F in $\mathcal{B}c$ is separated and hence isomorphic to $\widehat{\mathrm{I}}(\widehat{\mathrm{L}}(F))$. Combining this fact with Proposition 2.6, we find that the sequence

$$\cdots \to R_1(J,\widehat{\mathrm{I}}\circ F) \to R_0(J,\widehat{\mathrm{I}}\circ F) \to \widehat{\mathrm{I}}(\widehat{\mathrm{L}}(F)) \to 0$$

is strictly exact in $\mathcal{B}c$. Now, Corollary 5.7 entails that the sequence

$$\cdots \to R_1(J,F) \to R_0(J,F) \to \widehat{\mathrm{L}}(F) \to 0$$

is strictly exact in $\widehat{\mathcal{B}c}$ and hence that F is $\widehat{\mathrm{L}}$ acyclic. □

Proposition 5.14. *The functor*

$$\widehat{\mathrm{I}} : \widehat{\mathcal{B}c} \to \mathcal{B}c$$

is strictly exact and gives rise to a functor

$$\widehat{\mathrm{I}} : D^*(\widehat{\mathcal{B}c}) \to D^*(\mathcal{B}c) \qquad * \in \{+, -, b, \emptyset\}.$$

The functor

$$\widehat{\mathrm{C}}\mathrm{pl} : \mathcal{B}c \to \widehat{\mathcal{B}c}$$

is cokernel preserving and has bounded cohomological dimension. In particular, it has a left derived functor

$$\mathrm{L}\widehat{\mathrm{C}}\mathrm{pl} : D^*(\mathcal{B}c) \to D^*(\widehat{\mathcal{B}c}) \qquad * \in \{+, -, b, \emptyset\}.$$

and

$$\mathrm{L}\widehat{\mathrm{C}}\mathrm{pl} \simeq \mathrm{L}\widehat{\mathrm{L}} \circ \mathrm{CS}.$$

Moreover,

$$\mathrm{RHom}_{\widehat{\mathcal{B}c}}(\mathrm{L}\widehat{\mathrm{C}}\mathrm{pl}(E), F) \simeq \mathrm{RHom}_{\mathcal{B}c}(E, \widehat{\mathrm{I}}(F)).$$

and

$$\mathrm{L}\widehat{\mathrm{C}}\mathrm{pl} \circ \widehat{\mathrm{I}} \simeq \mathrm{id}\,.$$

In particular, $D^*(\widehat{\mathcal{B}c})$ may be identified with a full subcategory of $D^*(\mathcal{B}c)$ and $\mathcal{LH}(\widehat{\mathcal{B}c})$ may be identified with a full subcategory of $\mathcal{LH}(\mathcal{B}c)$.

Proof. The fact that $\widehat{\mathrm{I}}$ is strictly exact follows directly from Corollary 5.7.

The functor $\widehat{\mathrm{C}}\mathrm{pl}$ having a right adjoint is clearly cokernel preserving. Since $\widehat{\mathrm{C}}\mathrm{pl} \simeq \widehat{\mathrm{L}} \circ \mathrm{CS}$ and CS is exact, to prove the formula

$$\mathrm{L}\widehat{\mathrm{C}}\mathrm{pl} \simeq \mathrm{L}\widehat{\mathrm{L}} \circ \mathrm{CS}$$

it is sufficient to show that any object E of $\mathcal{B}c$ is a strict quotient of an object F for which $\mathrm{CS}(F)$ is $\widehat{\mathrm{L}}$-acyclic. We claim that we can take

$$F = \bigoplus_{B \in \tilde{\mathcal{B}}_E} E_B.$$

As a matter of fact, there is a canonical strict epimorphism

$$\bigoplus_{B \in \tilde{\mathcal{B}}_E} E_B \to E.$$

Moreover,

$$\mathrm{CS}(\bigoplus_{B\in\tilde{\mathcal{B}}_E} E_B) \simeq \mathcal{I}nd(\widehat{\cdot})(\mathrm{S}(\bigoplus_{B\in\tilde{\mathcal{B}}_E} E_B)) \simeq \bigoplus_{i\in I} \text{“}\widehat{E}_i\text{”}$$

and such an object is $\widehat{\mathrm{L}}$-acyclic.

Since $\widehat{\mathrm{I}}$ is stricly exact, Proposition 5.11 entails immediately that $\widehat{\mathrm{C}}\mathrm{pl}(P)$ is a projective object of $\widehat{\mathcal{B}c}$ is P is a projective object of $\mathcal{B}c$. To prove the derived adjunction formula it is thus sufficient to replace E by a projective resolution and apply Proposition 5.11 once more.

For any object E of $\widehat{\mathcal{B}c}$, $\mathrm{CS}(\widehat{\mathrm{I}}(E))$ is essentially monomorphic since

$$\mathrm{CS}(\widehat{\mathrm{I}}(E)) \simeq \text{“}\varinjlim_{B\in\tilde{\mathcal{B}}'_E}\text{”}\, E_B.$$

Thanks to part (c) of Proposition 5.13, it follows that $\mathrm{CS}(\widehat{\mathrm{I}}(E))$ is $\widehat{\mathrm{L}}$-acyclic. Therefore,

$$\mathrm{L}\widehat{\mathrm{C}}\mathrm{pl}(\widehat{\mathrm{I}}(E)) \simeq \mathrm{L}\widehat{\mathrm{L}}(\mathrm{CS}(\widehat{\mathrm{I}}(E))) \simeq \widehat{\mathrm{L}}(\mathrm{CS}(\widehat{\mathrm{I}}(E))) \simeq \widehat{\mathrm{C}}\mathrm{pl}(\widehat{\mathrm{I}}(E)) \simeq E.$$

□

Working as for $\mathcal{B}c$ and $\widehat{\mathcal{B}c}$, the results of this section easily entail the two following propositions.

Proposition 5.15. *The functor*

$$\widehat{\mathrm{S}} : \widehat{\mathcal{B}c} \to \mathcal{I}nd(\mathcal{B}an)$$
$$E \mapsto \text{“}\varinjlim_{B\in\tilde{\mathcal{B}}_E}\text{”}\, E_B$$

has as left adjoint the functor

$$\widehat{\mathrm{L}} : \mathcal{I}nd(\mathcal{B}an) \to \widehat{\mathcal{B}c}$$
$$\text{“}\varinjlim_{i\in I}\text{”}\, E_i \mapsto \varinjlim_{i\in I} E_i.$$

Moreover,

$$\widehat{\mathrm{L}} \circ \widehat{\mathrm{S}} = \mathrm{id}_{\widehat{\mathcal{B}c}}$$

and for any object F of $\mathcal{I}nd(\mathcal{B}an)$ the following conditions are equivalent:

(i) the canonical morphism $F \to \widehat{\mathrm{S}} \circ \widehat{\mathrm{L}}(F)$ is an isomorphism;

(ii) F is the essential image of $\widehat{\mathrm{S}}$;

(iii) F *is essentially monomorphic.*

In particular, $\widehat{\mathcal{B}c}$ *is equivalent to the full subcategory of* $\mathcal{I}nd(\mathcal{B}an)$ *formed by essentially monomorphic objects.*

Proposition 5.16. *(a) The functor*

$$\widehat{\mathrm{S}} : \widehat{\mathcal{B}c} \to \mathcal{I}nd(\mathcal{B}an)$$

is strictly exact and induces a functor

$$\widehat{\mathrm{S}} : D^*(\widehat{\mathcal{B}c}) \to D^*(\mathcal{I}nd(\mathcal{B}an)) \qquad (* \in \{-, +, b, \emptyset\}).$$

(b) The functors $\widehat{\mathrm{S}}$ *and* $\mathrm{L}\widehat{\mathrm{L}}$ *are quasi-inverse equivalences of categories. They induce an equivalence of categories*

$$\mathcal{LH}(\widehat{\mathcal{B}c}) \simeq \mathcal{LH}(\mathcal{I}nd(\mathcal{B}an)).$$

A Essentially monomorphic ind-objects

Definition A.1. Let $\mathcal{E}$ be a quasi-abelian category and let $E : \mathcal{I} \to \mathcal{E}$ be an ind-object of $\mathcal{E}$. We say that E is *monomorphic* (resp. *strictly monomorphic*) if for any $\alpha : i \to i'$ of $\mathcal{I}$ the morphism

$$E(\alpha) : E(i) \to E(i')$$

is a monomorphism (resp. strict monomorphism) of $\mathcal{E}$. An ind-object isomorphic to a monomorphic (resp. strictly monomorphic) ind-object is said to be *essentially monomorphic* (resp. *essentially strictly monomorphic*).

Proposition A.2. *Let* $\mathcal{E}$ *be a quasi-abelian category and let* $E : \mathcal{I} \to \mathcal{E}$ *be an ind-object of* $\mathcal{E}$*. Then,*

(a) *the following conditions are equivalent :*

(i) E *is essentially monomorphic;*

(ii) *for any* $i \in \mathcal{I}$*, there is* $\alpha : i \to i'$ *such that for any* $\alpha' : i' \to i''$ *the canonical strict monomorphism*

$$\operatorname{Ker} E(\alpha) \to \operatorname{Ker} E(\alpha' \circ \alpha)$$

is an epimorphism (and hence an isomorphism);

(iii) for any $i \in \mathcal{I}$, there is $\alpha : i \to i'$ such that for any $\alpha' : i' \to i''$ the canonical strict epimorphism

$$\operatorname{Coim} E(\alpha) \to \operatorname{Coim} E(\alpha' \circ \alpha)$$

is a monomorphism (and hence an isomorphism).

(b) the following conditions are equivalent :

(i) E is essentially strictly monomorphic;

(ii) for any $i \in \mathcal{I}$, there is $\alpha : i \to i'$ such that for any $\alpha' : i' \to i''$ the canonical epimorphism

$$\operatorname{Im} E(\alpha) \to \operatorname{Im} E(\alpha' \circ \alpha)$$

is a strict monomorphism (and hence an isomorphism).

Proof. (a) (i) $\Longrightarrow$ (ii). Assume $u : E \to F$ is an isomorphism of the ind-object $E : \mathcal{I} \to \mathcal{E}$ with a monomorphic ind-object $F : \mathcal{J} \to \mathcal{E}$ and denote $v : F \to E$ its inverse. For any $i \in \mathcal{I}$, there is $j \in \mathcal{J}$ and a representant

$$E(i) \xrightarrow{u_{ji}} F(j)$$

of u. Since $v \circ u = \mathrm{id}_E$, there is a representant $v_{i'j} : F(j) \to E(i')$ of v such that $v_{i'j} \circ u_{ji} = E(\alpha)$ where $\alpha : i \to i'$ is a morphism of $\mathcal{I}$. Consider a morphism $\alpha' : i' \to i''$ of $\mathcal{I}$. Since $E(\alpha') \circ v_{i'j} : F(j) \to E(i'')$ represents v and $u \circ v = \mathrm{id}_F$, we can find a representant $u_{j'i''} : E(i'') \to F(j')$ of u such that

$$u_{j'i''} \circ E(\alpha') \circ v_{i'j} = F(\beta)$$

where $\beta : j \to j'$ is a morphism of $\mathcal{J}$. Now, let $h : X \to E(i)$ be a morphism of $\mathcal{E}$ such that

$$E(\alpha' \circ \alpha) \circ h = 0.$$

Since

$$u_{j'i''} \circ E(\alpha' \circ \alpha) = u_{j'i''} \circ E(\alpha') \circ E(\alpha) = u_{j'i''} \circ E(\alpha') \circ v_{i'j} \circ u_{ji} = F(\beta) \circ u_{ji}$$

it follows that

$$F(\beta) \circ u_{ji} \circ h = 0.$$

Using the fact $F(\beta)$ is a monomorphism, we see that $u_{ji} \circ h = 0$. Therefore

$$v_{i'j} \circ u_{ji} \circ h = E(\alpha) \circ h = 0.$$

This shows that $\operatorname{Ker} E(\alpha) \simeq \operatorname{Ker} E(\alpha' \circ \alpha)$.

(ii) $\Longrightarrow$ (iii). This follows from the definition of the coimage.

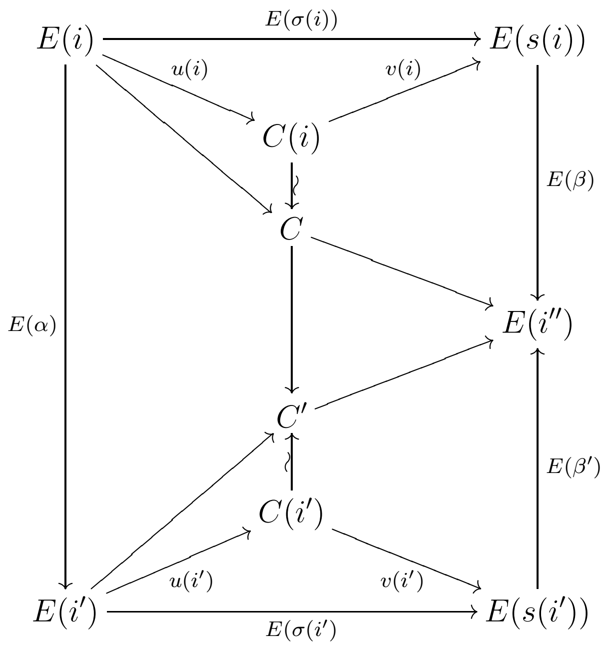
(iii) $\Longrightarrow$ (i). For any $i \in \mathcal{I}$, select a $\sigma(i) : i \to s(i)$ such that for any $\sigma' : s(i) \to i'$,

$$\operatorname{Coim} E(\sigma(i)) \simeq \operatorname{Coim} E(\sigma' \circ \sigma(i)).$$

Set $C(i) = \operatorname{Coim} E(\sigma(i))$ and denote $u(i) : E(i) \to C(i)$ and $v(i) : C(i) \to E(s(i))$ the canonical morphisms. Let $\alpha : i \to i'$ be a morphism of $\mathcal{I}$. Using the fact that $\mathcal{I}$ is filtering, it is possible to find $\beta : s(i) \to i''$, $\beta' : s(i') \to i''$ such that

$$E(\beta) \circ E(\sigma(i)) = E(\beta') \circ E(\sigma(i')) \circ E(\alpha).$$

Denote C and C' the coimages of $E(\beta) \circ E(\sigma(i))$ and $E(\beta') \circ E(\sigma(i'))$. We get the following commutative diagram of canonical morphisms



This gives us a monomorphism $C(\alpha) : C(i) \to C(i')$ which is easily seen not to depend on β or β'. Moreover, $C(\alpha' \circ \alpha) = C(\alpha') \circ C(\alpha)$ for any $\alpha' : i' \to i''$ in $\mathcal{I}$ and we get a morphism of functors

$$u : E \to C.$$

Moreover, we know that the canonical morphism

$$C \to E(i'')$$

is a monomorphism. It follows that the morphism $C \to C'$ is monomorphic. Therefore, $C(\alpha) : C(i) \to C(i')$ is also a monomorphism and C is a monomorphic ind-object.

Applying the canonical morphism

$$r_{s(i),i} : \operatorname{Hom}_{\mathcal{E}}(C(i), E(s(i))) \to \varinjlim_{j \in \mathcal{I}} \operatorname{Hom}_{\mathcal{E}}(C(i), E(j))$$

to $v(i)$ for any $i \in I$, we get a family

$$(r_{s(i),i}(v(i)))_{i \in I}$$

of elements of

$$\varinjlim_{j \in \mathcal{I}} \operatorname{Hom}(C(i), E(j)).$$

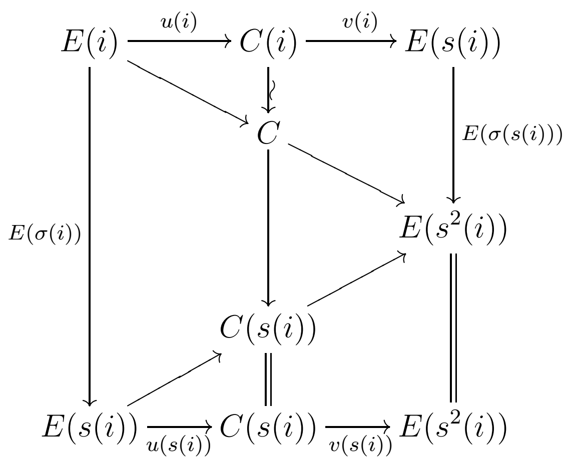
Since the commutative diagram above shows that

$$[\varinjlim_{j \in \mathcal{I}} \operatorname{Hom}(C(\alpha), \operatorname{id}_{E(j)})](r_{s(i'),i'}(v(i'))) = r_{s(i),i}(v(i)).$$

this family defines a morphism

$$v : C \to E$$

in $\mathcal{I}nd(\mathcal{E})$. Clearly, $v \circ u = \operatorname{id}_E$. Let us show that $u \circ v = \operatorname{id}_C$. Using the definition of $C(\alpha)$ for $\alpha = \sigma(i)$, we get the commutative diagram



where $C(\sigma(i))$ is represented by the composition of the second column of vertical arrows. It follows that

$$v(s(i)) \circ C(\sigma(i)) = E(\sigma(s(i))) \circ v(i) = v(s(i)) \circ u(s(i)) \circ v(i)$$

and since $v(s(i))$ is injective, we see that

$$u(s(i)) \circ v(i) = C(\sigma(i)).$$

Hence, $u \circ v = \mathrm{id}_C$. This concludes the proof of (a).

(b) (i) $\Longrightarrow$ (ii). Proceeding as in (a), for any $i \in \mathcal{I}$, we find $\alpha : i \to i'$ such that for any $\alpha' : i' \to i''$ we get a commutative diagram of the form

$$\begin{tikzcd} E(i) \arrow[r, "u_{ji}"] \arrow[rr, bend right, "E(\alpha)"'] & F(j) \arrow[r, "v_{i'j}"] \arrow[rrr, bend left, "F(\beta)"] & E(i') \arrow[r, "E(\alpha')"] & E(i'') \arrow[r, "u_{j'i''}"] & F(j') \end{tikzcd}$$

where $\beta : j \to j'$ is a morphism of $\mathcal{J}$ and $F(\beta) : F(j) \to F(j')$ is a strict monomorphism. It follows that $v_{i'j}$ is a strict monomorphism. In the canonical commutative diagram

$$\begin{tikzcd} \operatorname{Im} E(\alpha) \arrow[rr] \arrow[dr] & & \operatorname{Im}(v_{i'j}) \arrow[dl] \\ & E(i') & \end{tikzcd}$$

the diagonal arrows are strict monomorphisms. Hence, there is a canonical strict monomorphism

$$\operatorname{Im} E(\alpha) \to F(j).$$

Using the commutative diagram

$$\begin{tikzcd} \operatorname{Im} E(\alpha) \arrow[rrr] \arrow[dr] \arrow[dd] & & & \operatorname{Im} E(\alpha') \arrow[dl] \\ & E(i') \arrow[r] & E(i'') \arrow[dr] & \\ F(j) \arrow[ur] \arrow[rrr] & & & F(j') \end{tikzcd}$$

one sees easily that $\operatorname{Im} E(\alpha) \to \operatorname{Im} E(\alpha')$ is a strict monomorphism.

(ii) $\Longrightarrow$ (i). We may proceed entirely as in ((a), (iii) $\Longrightarrow$ (i)) replacing coimages by images. □

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